

Teaching Continuum Mechanics in a Mechanical Engineering Program

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1. Introduction

Continuum mechanics is a branch of mechanics that deals with the analysis of the kinematics and the mechanical behavior of materials modeled as a continuum. Modeling an object as a continuum assumes that the substance of the object completely fills the space it occupies, which is a simplifying assumption for analysis purposes. As emphasized by Mase [1], continuum mechanics is the fundamental basis upon which several graduate engineering courses are founded. These courses include elasticity, plasticity, viscoelasticity, and fluid mechanics. Gollub [2] also demonstrated the important role that continuum mechanics plays in contemporary physics. Therefore, it is beneficial to teach the principles of continuum mechanics to first-year graduate students or upper-level undergraduates to provide them with the necessary background in continuum theory so that they can readily pursue a formal course in any of the aforementioned subjects.

Due to its importance in engineering education, continuum mechanics has been built into the engineering curriculum and taught at many prominent universities, such as MIT, Texas A&M University, Carnegie Mellon, etc. Lagoudas et al. [3] demonstrated the significance of teaching continuum mechanics in their engineering program and designed a core continuum mechanics course for sophomore students at Texas A&M. In that course, conservation laws and fundamental concepts of continuum mechanics were taught using computer tools [4]. This paper describes the design of the course, Continuum Mechanics, for the Mechanical Engineering (ME) program at the University of Louisiana at Lafayette (UL Lafayette). Compared to similar courses taught in other universities, this course emphasizes the application of mathematics in formulating and solving fundamental equations of fluid and solid mechanics, as detailed in the course syllabus.

The paper is organized as follows. Section 2 describes the objectives of this course and briefly explains how the course objectives correlate with the ME program objectives.

Section 3 presents the detailed methodology and approach of teaching this course, which include topics (concepts, principles, laws, and equations), class organization, and evaluation instruments. Section 4 discusses the outcomes of teaching this course at the University of Louisville and indicates the modification of course structure since then. Finally, this paper is concluded and ended with section 5.

2. Course Objectives

Continuum Mechanics is a three-credit course which emphasizes the mathematics and analysis methods used in the study of the behavior of a continuous medium. The course is designed to be taken by first-year graduate students of the ME department at UL Lafayette. The primary objectives of this course are:

1. To study the conservation principles in the mechanics of continua and formulate the equations that describe the motion and mechanical behaviors of continuum materials, and
2. To present the applications of these equations to simple problems associated with solid and fluid mechanics.

As mentioned before, this course is a problem solving course which focuses on a mathematical study of mechanics of the idealized continuum mediums. As stated by Petroski [5] and Reddy [6], undergraduate mathematics plays an important role in continuum mechanics research and the continuous material behavior can be precisely described using modern algebra. As indicated from the course description, a fundamental basis in differential equations, mechanics of materials, and fluid mechanics are required before taking this course.

As stipulated by the Accreditation Board for Engineering and Technology (ABET) [7], the Mechanical Engineering Department has a set of eleven educational objectives which are to be satisfied by the curriculum. This course supports each of those objectives while emphasizing the following:

- Fundamentals. An ability to apply knowledge of mathematics, science, and engineering in the field of mechanical engineering.

Abstract

This paper introduces a graduate course, continuum mechanics, which is designed for and taught to graduate students in a Mechanical Engineering (ME) program. The significance of continuum mechanics in engineering education is demonstrated and the course structure is described. Methods used in teaching this course such as topics, class organization, and evaluation instruments are explained. Based on the student learning outcomes and feedbacks, the course objectives were achieved. This paper shows that ME program objectives are well supported by this course.

Keywords: continuum mechanics, mechanical engineering, first-year graduate students

- **Problem Solving.** An ability to identify, formulate and solve problems in the field of mechanical engineering.
- **Continuing Education.** A recognition of the need for, and an ability to engage in, life-long learning in the field of mechanical engineering.
- **Engineering Practice.** An ability to use the techniques, skills, and modern tools necessary for the practice of mechanical engineering.

3. Methodology

This course has a well-defined schedule and syllabus so that the instructor will cover various topics in the class, which is required by the breadth of the material and subject matter. Visual and hands-on learning techniques will be used wherever possible by the instructor of this course. In many cases, the ME students who take this course may have different backgrounds in mathematics. Students will be able to use class time and available office hours to discuss special topics with instructor. Descriptions of a course syllabus are presented. As reflected from the syllabus, mathematical methods and fundamental principles and laws play an important role in this class.

Course Syllabus

Continuum Mechanics

Credit Hours: 3

Course Goals

The goal of this course is to emphasize the formulation of problems in mechanics along with the fundamental principles that underlie the governing differential equations and boundary conditions.

Prerequisites by Topic

Differential equations; mechanics of materials; fluid mechanics

Text Book

Reading and homework assignments refer to the required textbook:
T. J. Chung, *General Continuum Mechanics*, 2007, Cambridge University Press, New York, NY, USA.

Organization

This class is organized into three 50 minutes sessions per week devoted to lecture/discussion and problem solving. The difficulty of this course is such that a minimum weekly commitment of 8-10 outside study hours will be required.

Quizzes

Daily quizzes will be given, typically over the previous reading assignment or the material covered in the last class. Missed quizzes cannot be made up.

Exams

Three closed-notes, closed-book exams will be given throughout the semester, which include two midterm exams and one final exam.

Grading System

Homework Assignments	20%
Quizzes	15%
Midterm Exams	2 @ 20%
Final Exam	25%
Total	100%

Final grades will be calculated according to an absolute scale:

≤ 59 F

Figure 1.

60-62 D-	63-67 = D	68-69 D+
70-72 C-	73-77 = C	78-79 C+
80-82 B-	83-87 = B	88-89 B+
90-92 A-	93-97 = A	98-100 A+

Topic/Activity

1. Introduction to continuum mechanics (2 classes)
 - What is continuum mechanics?
 - Main themes and assumptions of continuum mechanics.
2. Vectors, tensors and essential mathematics (6 classes)
 - Vectors and tensors
 - Tensor algebra; summation convention; Kronecker delta; permutation symbol; δ - δ identity.
 - Indicial notation; tensor product.
 - Matrix operation and linear algebra.
 - Transformation of Cartesian tensors.
 - Principal values (eigenvalue) and principal directions (eigenvector) of second order tensors.
 - Tensor fields and tensor calculus; partial differential operator; subscript comma.
 - Integral theorems of Gauss and Stokes
3. Stress principles (3 classes)
 - Body and surface forces; mass density.
 - Cauchy stress principles; Newton's 2nd and 3rd laws.
 - Stress vector; stress tensor; state of stresses.
 - Force and moment equilibrium; stress tensor symmetry.
 - Stress transformation laws.
4. Principal stresses and principal axes (3 classes)
 - Principal stresses and principal stress directions.
 - Maximum and minimum stress values.
 - Mohr's circle for stress.
 - Plane stress.
 - Spherical stress; deviator stress states.
 - Octahedral shear stress.
5. Analysis of deformation (3 classes)
 - Particles; configurations; deformation and motion.
 - Material and spatial coordinates; find velocity and acceleration for a given motion; Jacobian matrix.
 - Lagrangian and Eulerian descriptions.
 - Displacement field.
 - Material derivative.
 - Deformation gradients, Lagrangian and Eulerian finite strain tensors; Cauchy deformation tensor.
6. Velocity fields and compatibility conditions (3 classes)
 - Infinitesimal deformation theory; strain transformation; strain compatibility equations; normal strain and shear strain; cubical dilatation.
 - Calculate stretch ratios.
 - Rotation tensors and stretch tensors; polar decomposition.
 - Velocity gradient; rate of deformation and vorticity.
7. Fundamental law and equations (5 classes)
 - Material derivative of line elements, areas, and volumes; isochoric motion.
 - Balance laws; field equation; constitutive equation.
 - Material derivative of line, area, and volume integrals.
 - Conservation of mass; continuity equation.
 - Linear momentum principle; equation of motion and equilibrium equation.
 - Piola-Kirchhoff stress tensors; Lagrangian equation of motion.
 - Principle of angular momentum.
 - Conservation law of energy; energy equation.

Figure 1. (continued)

- General constitutive equations.
- 8. Linear elasticity (6 classes)
 - Elasticity; Hooke's law; strain energy.
 - Hooke's law for isotropic media; elastic constants.
 - Elastic symmetry; Hooke's law for anisotropic media.
 - Isotropic elastostatics and elastodynamics; principle of superposition.
 - Plane elasticity; difference between plane stress and plain strain.
 - Linear thermoelasticity.
 - Airy stress function and bi-harmonic equation.
- 9. Linear viscoelasticity (4 classes)
 - Elastic response of solids and viscous flow of fluids.
 - Linear viscoelastic materials; constitutive equations in linear differential operator form.
 - Construction of viscoelastic models; Maxwell model; Kelvin model; generalized model; 3-parameter solid and fluid model.
 - Creep and relaxation; stress relaxation function; Kelvin behavior and Maxwell behavior.
 - Principle of superposition; hereditary integrals; distortional and dilatational responses.
 - Harmonic loadings; complex modulus and complex compliance.
- 10. Classical fluids (2 classes)
 - Viscous stress tensor; Stokesian and Newtonian fluids.
 - Basic viscous flow equations; Navier Stokes equation.
 - Barotropic fluids, incompressible fluids, and frictionless fluids.
 - Steady flow, irrotational flow, and potential flow.
 - Bernoulli equation and Kelvin's theorem.
- 11. Nonlinear elasticity and review (3 classes)
 - Molecular approach to rubber elasticity; rubber stress; resilience.
 - Neo-Hookian material; Mooney-Rivlin model; Ogden material.
- 12. Examinations (2 classes and 2.5 hour final exam)

Figure 1. (continued)

4. Examples

Two examples are presented here to demonstrate the advantages of continuum mechanics in solving problems in fluid and solid mechanics.

Example 1. Fluid Mechanics Problem

An incompressible Newtonian fluid maintains a steady flow under the action of gravity down an inclined plane of slope β . If the thickness of the fluid is perpendicular to the plane h and the pressure on the free surface is $p = p_0$ (a constant), determine the pressure and velocity fields for this flow.

Solution: Assume $v_1 = v_3 = 0$, $v_2 = v_2(x_2, x_3)$. By the continuity equation for incompressible flow, $v_{i,i} = 0$. Hence, $v_{2,2} = 0$ and $v_2 = v_2(x_3)$. Thus, the rate of deformation tensor has components $D_{23} = D_{32} = 1/2(\partial v_2 / \partial x_3)$ and all others equal to zero.

The Newtonian constitutive equation is given in this case by

$$s_{ij} = -pd_j + 2m^*D_j$$

from which we calculate

$$[s_{ij}] = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & m^* \left(\frac{\partial v_2}{\partial x_3} \right) \\ 0 & m^* \left(\frac{\partial v_2}{\partial x_3} \right) & -p \end{bmatrix}$$

Because gravity is the only body force,

$$b = g(\sin \mathbf{b}\hat{e}_2 - \cos \mathbf{b}\hat{e}_3)$$

the equations of motion having the steady flow form: $\sigma_{ij,j} + \rho b_i = \rho v_j v_{i,j}$
resulting in component equations:

$$(\text{for } i = 1) -p_{,1} = 0$$

$$(\text{for } i = 2) -p_{,2} + \mu^*(\partial^2 v_2 / \partial x_3^2) + \rho g \sin \beta = 0$$

$$(\text{for } i = 3) -p_{,3} - \rho g \cos \beta = 0$$

Integrating the last of these gives: $p = (\rho g \cos \beta)x_3 + f(x_2)$

where $f(x_2)$ is an arbitrary function of integration. At the free surface ($x_3 = h$), $p = p_0$, we have: $f(x_2) = p_0 - \rho g \cos \beta$

and thus: $p = p_0 + (\rho g \cos \beta)(x_3 - h)$, which describes the pressure in the fluid.

Next, by integrating the middle equation above (for $i = 2$) twice with respect to x_3 , we obtain:

$$v_2 = \frac{-\rho g \sin \beta}{2\mu^*} x_3^2 + ax_3 + b$$

with a and b constants of integration. But from the boundary conditions: 1) $v_2 = 0$ when $x_3 = 0$, therefore $b = 0$; 2) $\sigma_{23} = 0$ when $x_3 = h$, therefore $a = (\rho g h / \mu^*) \sin \beta$.

Finally, therefore, from the equation for v_2 we have by the substitution of $a = (\rho g h / \mu^*) \sin \beta$, we obtain:

$$v_2 = \frac{\rho g \sin \beta}{2\mu^*} (2h - x_3)x_3$$

Finish example 1.

Example 2. Solid Mechanics Problem

Consider a special stress function having the form

$$\Phi^* = B_2 x_1 x_2 + D_4 x_1 x_2^3$$

Show that this stress function may be adapted to solve for the stresses in an end-loaded cantilever beam shown in the Fig. 1. Assume the body forces are zero for this problem.

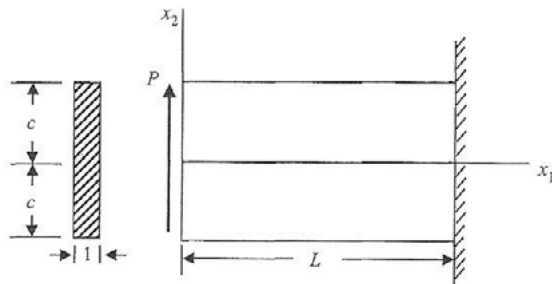


Figure 1. Cantilever beam loaded at the end by force P

Solutions: It is easily verified, by direct substitution, that $\nabla^4 \Phi^* = 0$. The stress components are directly computed as

$$\sigma_{11} = 6D_4 x_1 x_2^2; \quad \sigma_{22} = 0; \quad \sigma_{12} = -B_2 - 3D_4 x_2^2$$

These stress components are consistent with an end-loaded cantilever beam, and the constants B_2 and D_4 can be determined by considering the boundary conditions. In order for the top and bottom

surfaces of the beam to be stress free, σ_{12} must be zero at $x_2 = \pm c$. Using this condition, B_2 is determined in terms of D_4 as $B_2 = -3D_4c^2$. The shear stress is thus given in terms of single constant B_2

$$\sigma_{12} = -B_2 + (B_2x_2^2)/c^2$$

The concentrated load is modeled as the totality of the shear stress σ_{12} on the free end of the beam. Thus, the result of integrating this stress over the free end of the beam at $x_1 = 0$ yields the applied force P. in equation form

$$p = \int_{-c}^{+c} -B_2 + B_2 \frac{x_2^2}{c^2} dx_2$$

where the minus sign is required due to the sign convention on shear stress. Carrying out the integration, we have $B_2 = 3P/4c$ so that stress components may now be written as:

$$\sigma_{11} = (-3P/2c^3)x_1x_2; \quad \sigma_{22} = 0; \quad \sigma_{12} = (-3P/4c)(1 - x_2^2/c^2)$$

But for this beam, the plane moment of inertial of the cross section is $I = 2c^3/3$ so that now

$$\sigma_{11} = (-P/I)x_1x_2; \quad \sigma_{22} = 0; \quad \sigma_{12} = (-3P/2I)(c^2 - x_2^2)$$

in agreement with the results of elementary beam bending theory.

Finish example 2.

5. Discussion

This course was taught by the author to the first-year ME graduate students at University of Louisville in Fall of 2006. As an entry-level graduate course, Continuum Mechanics formed a substantial background for the students and prepared them for further graduate courses, such as Advanced Engineering Mathematics (required by all PhD students in Engineering School), Experimental Stress Analysis, Computational Methods in Fluid Flow and Heat Transfer, Mechanics of Biomaterials, and Advanced Fluid Mechanics.

Six students enrolled in this class and by the end of that semester, favorable outcomes were received either from the student feedback or from the course evaluation. The evaluation results (Table 1) reflected that this course is well designed, so that the students learned fundamental knowledge of continuum mechanics and mastered basic mathematic skills in studying the kinematics and mechanical behavior of the continuous mediums. The average score of the class is 82/100 with two students receiving an "A". After taking this course, five of the six students successfully passed the comprehensive PhD qualifying exam and are working for their Doctoral degrees at the University of Louisville. In the qualifying exam, eight problems were given based on the eight major areas in Mechanical Engineering. Three of them are tightly related to the materials taught in the Continuum Mechanics, which are Solid Mechanics, Fluid Mechanics, and Advanced Mathematics.

Table 1. Selected course evaluation reports

Query	<i>Strongly agree</i>	<i>Agree</i>	<i>Undecided</i>	<i>Disagree</i>	<i>Strongly disagree</i>
<i>I learned a lot in this course</i>	5 (84%)	1 (16%)	0	0	0
<i>This course challenged me to think</i>	4 (67%)	2 (33%)	0	0	0
Summary	<i>Excellent</i>	<i>Good</i>	<i>Average</i>	<i>Fair</i>	<i>Poor</i>
<i>Overall, I would rate this course as</i>	5 (84%)	1 (16%)	0	0	0
<i>Overall, I would rate the effective-ness of this instructor as</i>	5 (84%)	1 (16%)	0	0	0

Student feedback was also exciting; it showed that the students had learned substantial principles from this class and gained a new appreciation for the study of mechanics. This appreciation aroused their interests in continuing to work for a Doctoral degree in the ME department. Selected comments from the students include:

"Amount of knowledge gained was tremendous. This class will be very helpful in my career."

"The class was well organized as always. Dr. Liu presents a caring, considerate and open attitude toward the students."

"This is one of the toughest classes, but Dr. Liu is extremely effective at communicating difficult concepts in the classroom."

"I have learned a lot from this class and the instructor has earned my respect!"

With the success of offering the Continuum Mechanics in University of Louisville, the same class can also be implemented into ME curriculum as a core graduate course at UL Lafayette with similar course structure. Due to its low class/laboratory requirements, no extra equipments and facilities are needed for this class. Also, the class organization is so flexible that it can be folded into a short class offered for the summer semester.

6. Conclusion

This paper describes the design of the course Continuum Mechanics for the ME graduate students. Its significance and high impact on engineering education and research have been fully demonstrated. The course structure is presented in the paper, from which it can be found that this course especially focuses on the understanding and using of advanced mathematics in the study of mechanics within continuous mediums, which makes it unique from other mechanics classes. As verified in the paper, this course closely aligns with the ME program, and major ABET program objectives are satisfied with the establishment of this course.

This course had been taught by the author in the ME department at University of Louisville and inspiring feedback was received from the students. Table 1 shows that 84% of students "strongly agreed" that they had learned a lot from this course, 67% students "strongly agreed" that this course had challenged them to think, and 84% students rated this course as "excellent." Consequently, the overall teaching evaluation score for this course was 1.23, which was markedly above the department average, 1.89, and the university average, 1.75.

From teaching this course, it was also found that if this course was provided for the first-year graduate students, it forms a solid background for further ME graduate courses and makes students competitive for their PhD programs. With its favorable effects in engineering graduate program, the author plans to introduce a similar course at UL Lafayette which will enrich course offerings in the ME department and the Engi-

neering college, thereby greatly enhancing the Engineering program in UL Lafayette.

References

1. Mase, G. T., and Mase, G. E. (1999). *Continuum Mechanics for Engineers*, 2nd Ed., Boca Raton: CRC Press.
2. Gollub, J. (2003). "Continuum Mechanics in Physics Education." *Physics Today*, December 2003, 10-11.
3. Lagoudas, D. C., Whitcomb, J. D., Miller, D. A., Lagoudas, M. Z., Shryock, K. J. (2000). "Continuum mechanics in a restructured engineering undergraduate curriculum", *International Journal of Engineering Education*, 16(4). pp. 301-314.
4. Lagoudas, D. C., Whitcomb, J. D., Miller, D. A., Lagoudas, M. Z. (1997). "Computer aided teaching of continuum mechanics to undergraduates", *Proceedings of ASEE Gulf-Southwest Annual Conference*, The University of Houston, March 23-25.
5. Petroski, H. J. (1973). "Mathematics in continuum mechanics", *International Journal of Mathematical Education in Science and Technology*, 4(3). pp. 227-231.
6. Reddy, J. N. (2008). *An Introduction to Continuum Mechanics*, New York: Cambridge University Press.
7. "2009-2010 Criteria for Accrediting Engineering Programs." Accreditation Board for Engineering and Technology (ABET), 12/1/2008.

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