The Development of Pedagogical Content Knowledge in First-Year Graduate Teaching Assistants

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Abstract

Our investigation is concerned with new teachers developing their ability to understand student thinking. We conducted individual interviews with graduate students teaching calculus for the first time, interviewing a representative sample of graduate students before and after their first teaching assignment. The interviews were transcribed and coded in order to understand the development of the mathematical knowledge for teaching in beginning teachers.

Introduction

Most university professors receive their first experience teaching undergraduates during their graduate school career, often with little or no training. Over the past several decades, graduate student training for teaching has evolved in some departments to much more structured pre-service and in-service training programs. The education research community has recently begun to investigate how mathematics graduate students and university faculty learn the craft of teaching in addition to how undergraduate students learn mathematics (Holton, 2001; Kung, 2010; Speer, Gutmann, & Murphy, 2005; Speer, Strickland, & Johnson, 2005). Moreover, veteran university faculty have developed materials for instructing graduate students in the craft of teaching undergraduate mathematics (Friedberg et al., 2001; Krantz, 1999; Mattuck, 1999; Rishel, 2000).

Most would agree that a sound knowledge of subject matter and a solid understanding of pedagogy are necessary for successful teaching; however, neither is a good predictor alone of effective teaching. Research has demonstrated that mathematical content knowledge alone is not a good predictor of teaching effectiveness (Beagle, 1979). Likewise, a knowledge of pedagogy—such as the ability to employ effective teaching strategies or organize a classroom—is not sufficient to successfully teach mathematics to undergraduates. One must also know the content of what one is teaching. A successful teacher will also possess the special type of mathematical knowledge required to follow student arguments, to foresee student difficulties with the subject matter, and to plan accordingly.

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If one is to be successful in the classroom, it is essential to understand student misconceptions, and to make clear explanations to one's students. To investigate the development of a new teacher's ability to understand student thinking in calculus and linear algebra, we interviewed seven graduate students before and after their first teaching assignment. The interviews were transcribed and coded for analysis. We recorded whether pedagogical content knowledge and the ability to apply mathematical knowledge to teaching increased during their first teaching experience. We also examined their beliefs on the role of teaching as part of their academic careers, and how these changed after having taught their first course.

Mathematical Knowledge for Teaching

While teachers must have a solid knowledge of the mathematics that they are teaching, success in the classroom also requires the teacher to be able to predict the different problem-solving strategies that students will employ, or the difficulties that they will encounter when dealing with a new mathematical concept. This type of knowledge is essential when dealing with student understanding, and allows a teacher to understand what their students are communicating in the classroom. Shulman (1986) and other researchers identified this type of knowledge as *pedagogical content knowledge* (PCK) in the 1980s. Hill, Ball, and Schilling have proposed a model for the mathematical knowledge for teaching (MKT) that refines Shulman's original model (Hill, Ball, & Schilling, 2008). In this new model, both subject matter knowledge and pedagogical content knowledge (PCK) are further decomposed (Figure 1). Subject matter knowledge is divided into the areas of common content knowledge (CCK), specialized content knowledge (SCK), and knowledge at the mathematical horizon. Common content knowledge is the mathematical knowledge used in other professions as well as in teaching. In the course of teaching mathematics, a teacher will occasionally have need of mathematical facts that they have not encountered in their formal study. These are examples of specialized content knowledge. As an example of SCK, we might consider the process of

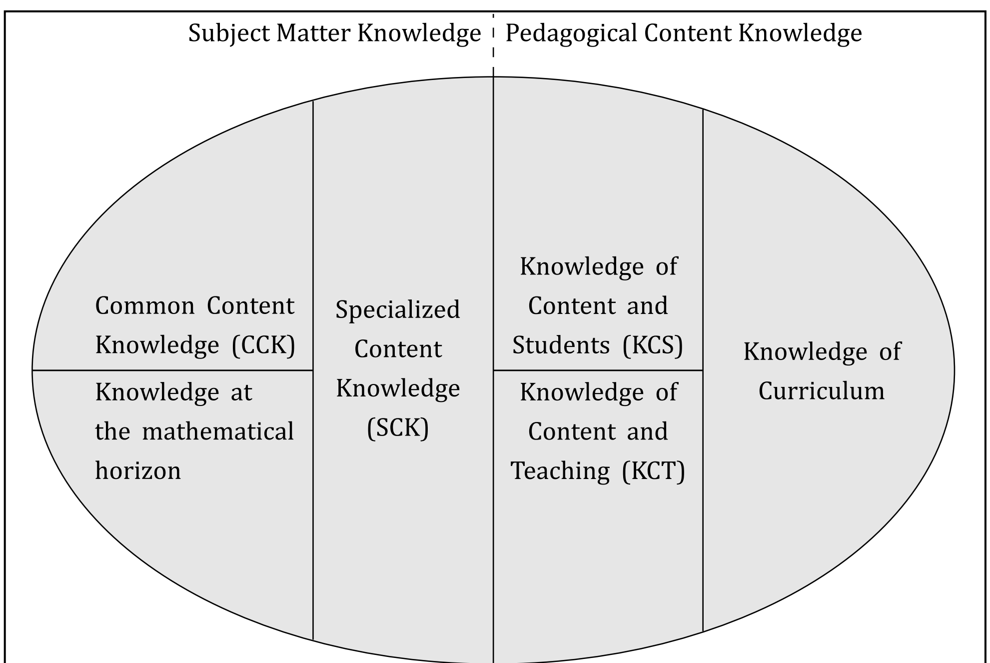

Figure 1. Domain map for mathematical knowledge for teaching

decomposing rational functions into partial fractions, an algebra skill taught in a first-year calculus course. For instance,

$$\frac{2x+1}{(x^2-1)(x^2+1)} = \frac{1}{4(x+1)} + \frac{3}{4(x-1)} - \frac{2x+1}{2(x^2+1)}.$$

Most instructors know this procedure; however, many have not studied the existence and uniqueness of the partial fraction decomposition. In fact, with the exception of Lang's *Algebra*, most graduate textbooks omit the proof of this fact or leave it as an exercise (Lang, 2005). Yet, in the calculus classroom, a good student might understand the procedure of partial fraction decomposition and still ask if the decomposition will always work.

A third category of subject matter knowledge is knowing what mathematics their students will encounter in future courses, or knowledge at the mathematical horizon. For example, an elementary school teacher needs to understand the importance of teaching the distributive law when teaching multiplication of whole numbers, since students will need to use this knowledge to succeed when they study algebra. The precalculus instructor must understand the importance of the student mastery of the functional notation needed for the study of calculus.

Pedagogical content knowledge can be divided into knowledge of content and students, knowledge of content and teaching, and knowledge of curriculum. Knowledge of content and students (KCS) is knowing how students will learn mathematics, what types of strategies that they will employ to solve problems, and the pitfalls that they will encounter. Knowledge of content and teaching (KCT) allows the teacher to devise strategies to guide student learning in the classroom. An instructor must also have a thorough knowledge of what mathematics is to be taught and in what order, which we shall call the knowledge of curriculum.

To help clarify our use of these terms, we will provide an illustration using as an example *l'Hôpital's rule* for limits:

If, $\lim_{x\to a} f(x) = 0,$ *and* $\lim_{x\to a} g(x) = 0,$

and $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L,$

, then $\lim_{x\to a} \frac{f(x)}{g(x)} = L.$

L'Hôpital's Rule is both a deep theorem and a powerful tool for computing limits of functions expressed in an indeterminate form. Calculus experts know that the theorem remains true when a, 0, or L is replaced by ∞. Students and instructors of a rigorous calculus or analysis course may know that the theorem is a consequence of the Cauchy Mean Value Theorem or Taylor's Theorem. Experts will also understand that l'Hôpital's rule can applied iteratively, as in:

$$\lim_{x\to 0}\frac{\sin x - x}{x^3} = \lim_{x\to 0}\frac{\cos x - 1}{3x^2} =$$
$$\lim_{x\to 0}\frac{-\sin x}{6x} = -\frac{1}{6},$$

but there is a subtlety in this derivation covering the fact that none of the equal signs are legitimate until the final limit is determined to exist. We classify these and related statements as subject matter knowledge.

On the other hand, experienced educators will be aware of the many pitfalls that students will face when learning l'Hôpital's rule for the first time. They will expect students to confuse the notion of a function failing to have a limit at a point and a limit expressed in indeterminate form—in other words, an indeterminate limit may still "exist." They will anticipate that armed with the sledgehammer of l'Hôpital's rule, students will attempt to use it on limits that are not in indeterminate form, arriving at wrong answers, as in:

$$\lim_{x\to 0^+}\frac{x}{\cos x} \overset{\text{not really}}{=} \lim_{x\to 0^+}\frac{1}{-\sin x} = -\infty$$

(this limit is not in an indeterminate form; the correct answer is found to be 0 by direct substitution). They will also understand that students are not as fluent in precalculus as instructors and will have difficulty remembering basic facts about transcendental functions such as cos(0)=1,ln(1)=0, and $\lim_{x\to 0^+}$ ln $x = -\infty$. Statements like these are examples of KCS. In order to avoid these pitfalls, experienced educators may use KCT to construct lessons on L'Hôpital's rule. They will provide examples to show how the conclusion of L'Hôpital's rule can fail if the hypotheses are not met, or show how to manipulate other indeterminate forms (for instance, $\infty - \infty$) into indeterminate quotients.

Methodology

Research Design and Data Collection

Our study investigated the pedagogical content knowledge required for teaching that graduate students acquire during their first teaching assignment. We used a qualitative approach to study the experiences of beginning graduate students and how they formulate the pedagogical knowledge that they use in the undergraduate calculus classroom (Creswell, 2007; Marshall & Rossman, 1999). We collected data through in-depth interviews with seven graduate students before and after their first teaching assignment. Each interview was about 20 to 40 minutes in length. All interviews were audio and videotaped. The videotapes were especially useful when participants wrote specific mathematical expressions or drew geometric figures, as it can be very difficult to analyze this type of data from voice recordings alone. When the interviews were completed, we transcribed each interview for coding and analysis. We followed standard methods for coding and analysis such as those described by Bogdan and Biklen (2006) or Miles and Huberman (1994).

Our coding categories included background, career plans, common content knowledge, specialized content knowledge, knowledge of content and students, and knowledge of content and teaching. We did not code for knowledge of curriculum or knowledge at the mathematical horizon.

- Following Speer and Wagner (2009), we defined common content knowledge (CCK) as the formal mathematical knowledge that mathematics faculty and graduate students have developed through study and/or research.
- Specialized content knowledge (SCK) was defined to be any mathematical knowledge needed for teaching that one does not ordinarily acquire though study or research.
- We defined knowledge of content and students (KCS) to be any knowledge of how students learn and do mathematics as well as what types of misconceptions that they might have.
- Knowledge of content and teaching (KCT) is how teachers combine mathematical knowledge with teaching to support student learning, such as devising examples and strategies for use in the classroom.

We collaborated closely to resolve any coding discrepancies. Furthermore, we distinguished between the pre- and post-teaching interviews.

We used the following questions to guide our initial interview with the graduate students.

- *Please tell us a little bit about yourself. Where are you from, what is your undergraduate background, and why are you interested in mathematics?*
- *After you graduate, what sort of position would you like to find?*
- *How do you see teaching as part of your future career plans? How does this balance with your research plans?*
- *What do you find challenging about teaching, as you prepare for your first teaching role?*

In addition, we gave each participant four different calculus problems and asked them to tell us what they would say to a student who requested help on one of these problems.

For our second interview with the participants, we used the following questions.

- *Now that you've had a chance to work with students, has your view of teaching or your career plans changed at all?*
- *What was your general view of the students and were there any unexpected surprises?*

We gave the participants four more calculus problems with the same instructions as before. Examples of these calculus problems can be found below.

Participants

Our sample was representative of many graduate programs in mathematics consisting of graduate students from Asia, Europe, and the United States. In addition to the two American students (Christopher and Sean), the two graduate students from Europe (Samantha and Amanda) had received their undergraduate education from U.S. universities and had some familiarity with the American university system.[1] None of the Asian students (Evan, Luis, and Tim) had attended American universities before entering graduate school. Both men and women were represented in our investigation. All were planning on careers in mathematical research and were seriously considering academia. They all felt that teaching undergraduates was extremely important; however, one graduate student did not have a good experience teaching and felt strongly about avoiding any research position that might include teaching duties.

Two participants did have some experience teaching during their undergraduate careers. Samantha had worked as a course assistant in several upper division courses leading discussion sessions, holding office hours, and grading homework. Christopher had a more in depth teaching experience as an undergraduate. He led problem-solving sessions for multivariable calculus and taught at several summer math camps for talented and gifted high school students.

Undergraduate students at the institution where our study took place are taught calculus and linear algebra in small sections, and the participants in our investigation were primary providers of instruction for these students. The graduate students in our study had all received pre-service training but none had taught an undergraduate course in the mathematics department before the initial round of interviews. A second interview was conducted with each of the participants after they had concluded their first teaching assignment.

Pre-service Education for Teaching Fellows

The calculus and linear algebra courses at the university where we conducted our investigation are primarily managed by a group of teaching faculty who are responsible for the first two years of mathematics education. The duties of the teaching faculty include teaching, graduate student teacher training, and undergraduate education in addition to their own research and scholarly activities.

Calculus and linear algebra courses are divided into sections, each with an average enrollment of 20–25 students. The course coordinator decides the syllabus, assignments, and grading policies for all sections and teaches at least one of the section of the course. The remaining sections are taught by graduate student teaching fellows (TFs), as well as postdoctorate and in some cases permanent faculty. The course co-ordinator and section instructors meet weekly to monitor progress, write the exams, and decide final course grades. This organization provides students with the benefits of an experienced faculty member managing the course and making the important decisions and a small classroom environment for instruction at the same time. The TFs gain valuable teaching experience, while at the same time receiving faculty and peer mentoring in teaching from the course coordinators and their fellow instructors.

Graduate students are fully supported during their first year of study and have no teaching obligations. In subsequent years, those without outside funding are required to teach one semester during each academic year. The pre-service program for teaching fellows typically begins before their first teaching year—either their first year of graduate study or their last year of outside funding.

Since the mid-1990s, TF training has consisted of a program known as the apprenticeship. Each graduate student apprentice is assigned to an experienced instructor or coach. (The investigators of this project did not interview graduate students who they have also coached during their apprenticeship.) Coaches are most often teaching faculty, but outstanding postdoctorate and experienced graduate student TFs have also served as coaches. The apprentice shadows the coach's class at least three times, and then teaches three regular classes to the students in the coach's section. The first of these classes is previewed in a dress rehearsal to student volunteers from the section, and the second class is videotaped and then reviewed by the coach and apprentice. After the final class taught by the apprentice, the coach makes a recommendation that the apprentice be given his or her own section of calculus, or that additional steps should be taken before the apprentice can teach independently.

Under the original apprentice program, graduate students who did not fare well under the apprenticeship were not always given the guidance necessary to meet those goals. To address this shortcoming, the teaching faculty began hosting additional events for the graduate TFs such as seminars on lesson planning and working with individual students. These seminars were well attended by teaching faculty, but participation among the target audience of novice teachers was low.

Presented with graduate students in need of additional training, and a series of teaching talks in need of participants, the teaching faculty expanded the apprentice program to include a teaching seminar series and make part of it mandatory for pre-service teaching fellows during the 2005–2006 academic year. The pre-service teaching seminar focused on basic aspects of teaching, such as how to prepare a lesson plan that would meet teaching goals, how to understand and react to questions, how to stand in front of a class and command respect, and what resources are available for students encountering difficulties. Specialists from the university's teaching center provided their expertise to help graduate students with public speaking skills and oral English when necessary. The teaching seminar stressed several levels of teaching activities. Participants would work one-on-one with students in the department's nightly homework help center. They would run exam workshops with groups of students, focusing on problem-solving strategies for upcoming tests. They would also undergo *microteaching* exercises, mock-teaching short lesson snippets in front of peers playing the roles of students (Allen & Eve, 1968; Allen, 1966).

Subsequent offerings of the pre-service teaching seminar included a deeper involvement with actual students outside of the classroom. In one session, graduate students interviewed the undergraduates to learn about their backgrounds, not only in mathematics but also in other academics and extracurricular activities. In other sessions, the undergraduates would observe and comment on microteaching sessions. The number of classroom observations increased, and the seminar participants would watch an experienced instructor's first class, as well as classes related to their own microteaching assignments. Graduate students must now complete the pre-service teaching seminar before they are allowed to apprentice.

Findings

Our analysis of the participant interviews was guided by the following questions.

1. How did the participants view the role of teaching as part of their career before and after their first teaching experience?
2. Does a participant's pedagogical content knowledge and awareness of the need for pedagogical content knowledge increase from before to after the first teaching experience?

In order to follow discussion thread involving the graduate students in our investigation, we refer the reader to Table 1.

View of Teaching

The participants were graduate students at a research institution, and they all considered mathematical research to be very important, but all acknowledged and embraced the importance of teaching. Luis expressed that his ideal position would be divided with 60% devoted to research and 40% devoted to teaching. Several TFs felt that the second half of the course that they were teaching went much smoother than the first half. The graduate students also reported that they learned how important it was to prepare for the classroom. Tim said, "I think that I do really need a lot of time to carefully prepare the material I have to teach. It makes things really different." He also pointed out how important it is for the instructor to respect their students and was critical of those TFs who did not demonstrate this respect.

All of the international students spoke excellent

TF	Background	Career Plans	Pre-Teaching PCK	Post-Teaching PCK
Amanda	From Eastern Europe. Undergraduate and masters degrees from U.S. universities.	Research position in academia or industry. No plans to teach.	Very limited awareness of the need for PCK	Did not seem to develop the ability to apply her mathematical knowledge to teaching situations.
Christopher	From U.S. Undergraduate degree from U.S. university.	Research and teaching position.	Teaching experience as an undergraduate. Aware of the need for PCK.	Developing the ability to apply his mathematical knowledge to teaching situations.
Evan	Undergraduate degree from an Asian university. Math and physics background.	Research and teaching position.	Limited awareness of the need for PCK.	Understands that some of his students lack fluency in pre-calculus.
Luis	Undergraduate degree from an Asian university. Math and physics background.	Research and teaching position.	Aware of the need for PCK.	Developing the ability to apply his mathematical knowledge to teaching situations.
Samantha	From Eastern Europe. Undergraduate degree from U.S. university.	Sees teaching as part of her career.	Limited teaching experience as an undergraduate and very limited awareness of the need for PCK.	Developed a deep understanding of the need for PCK and realized that she did not always know the source of her own mathematical knowledge.
Sean	From U.S. Undergraduate degree from U.S. university.	Enjoys teaching but uncertain about an academic career.	Some awareness of PCK. Acknowledged the influence of several of his own teachers.	Very aware of the need for PCK, but surprised at the mathematical knowledge of his students.
Tim	Undergraduate and masters degrees from an Asian university. Math and physics background.	Research and teaching position.	Very aware of the need for PCK and able to give multiple explanations for a particular question.	Very aware of the need for PCK and the need to carefully prepare for each class.

Table 1: Graduate Student Participants

English; however, the following statement by Tim summarizes the language difficulties that they encountered.

"I have a very big problem when I was teaching; it was a very big problem for me at first. When my students are asking some questions, sometimes I can't understand what they are asking. It is actually a very big problem for me at first. Sometimes a voice is very small, and sometimes it is really fast, like I can't understand."

TFs from countries other than the U.S. reported having difficulties that one might expect. For example, Tim was not used to the number of questions that students will ask in the typical American classroom.

"Well, yes. I think a major difference is that students here really like asking all kinds of questions. I think it's good. In China, sometimes students are like, 'I think it's maybe a silly question,' and maybe not ask. But here, whenever it's a silly question—or it's actually not a silly question, it'll really help you understand the material. Then students really like asking all kinds of questions. I think it's a little bit different."

Samantha, who had been an undergraduate at an American university, arrived at the realization of how important student questions can be in the classroom.

"I come from a system where you never ask questions, you know? It's taught in a very different way. And as much as I tried to change, I still don't ask so many questions. I mean, maybe I do now as a graduate student, but in undergrad I didn't ask so many questions. And I realized that—I mean, I've always thought that the professor doesn't like to have all that many questions. And it just sounds silly sometimes. And then when I taught, I realized that even the serious questions, I really wanted those questions, you know, just to be sure that everyone is—It was a very different perspective that I got."

All of the participants realized the importance of teaching, and all but one had a positive experience teaching their first course. Samantha enthusiastically summed up her experience in the following quote.

"It went great. I really loved it. I mean, I thought I'd like teaching, but it went better than I expected. I was nervous, but only for the first couple of classes. Then I really became comfortable with them. I had 29 students. And they were—I don't know. I felt that I had a really good section. So overall, they did great. And they were—they asked a lot of questions. They are pretty demanding. They really want to know things. And you can't just get away with stuff with them. There will definitely be at least one person who has something to say, you know? So I thought that was great. But I realized how much I love questions. I mean, whenever they were a little tired and they weren't asking so many questions, I felt sad, you know? It feels great when they have questions and you feel that they understand everything."

Amanda, the only graduate student who chose not to pursue a career involving teaching, said that teaching was important in the pre-teaching interview but that it was not a primary career goal of hers. After teaching a mathematical modeling course for biology students, she reported:

"One would expect that if we taught more about biology they would be happy. No. I actually talked with some students.... All they want to know is what to be on the exam and get out of there. If I actually had to do it again, I wouldn't spend so much time explaining biology in the paper, and I would just go straight to the models and talking about the maps and things like that."

When asked in the post-teaching interview how teaching would play a role in future career plans, Amanda responded, "by its absence."

Knowledge of Content and Students

Many of the TFs were surprised by the algebra and precalculus skills in some of their students. They also did not expect students with different levels of preparation to be in the same classroom. Sean remarked that some of the students confused basic algebraic facts, while others who had taken calculus in high school seemed to be overprepared.

> "I can think of maybe two or three students who had serious, serious math problems, like not understanding that x over 3 is $(1/3)x$, or that $x - y$ is the same thing as $x + (-y)$."

They also noticed that students who understood some of the basic precalculus facts had trouble combining two or more ideas. Tim remarked:

> "When I was teaching, students would really ask me sometimes some questions that I would never expect. I saw at first, for example, for log x times a constant. Everyone knows the derivative is $1/x$ times the constant. Then, I put some kind of extra constant, then people are very confused."

Sean remarked that there were many precalculus facts that he did not think about on a day-to-day basis—such as the values of trigonometric functions at zero, π, and $\pi/2$, or trigonometric identities. He was unprepared for student questions such as "Do we need to know this?" or "What's important to remember and what isn't?" He found that this phenomenon extended to calculus and that students had difficulty when asked to combine several ideas or techniques to solve a more complicated problem.

> "If they need to use the product rule alone, that's fine. If they need to use the product rule on the chain rule, that's fine. But if you need to use the product rule, the chain rule, and something else...."

Evan felt that most of the undergraduate students did in fact possess the necessary prerequisite mathematical knowledge to be successful in calculus, but that did not have ability to quickly access their knowledge.

> Q: Do you think it's that these basic facts about algebra and trigonometry is that they don't know them, or that they just lack the necessary fluency?
> A: Oh, it's just lack.
> Q: Lack fluency?
> A: Yeah. They're just slow, yeah. Because if I—Whenever I rewrote it slowly and tell—For example, I can just subtract this and divide this and multiply this, then the first line becomes the second line. But then they can't follow it. But if I just do it line by line slowly.

Building on PCK and Anticipating Student Difficulties

All of the participants in our study realized the need for pedagogical content knowledge even before their first teaching assignment. Although they had pre-service training and had successfully completed their apprenticeship, they did not have the extensive classroom experience needed to develop their pedagogical content knowledge. During the pre-teaching interview, we asked the following question.

> Question. Suppose that $f(1)=3$ and $f'(x)<2$ for all $x \in [0,5]$ One of your students is trying to decide if $f(4)\geq 9$. What would you say to the student?

Although the TFs gave correct explanations, these explanations were not always the most elementary or the easiest to understand. For example, several gave explanations using the Fundamental Theorem of Calculus or the Mean Value Theorem. However, a few of the TFs gave graphical explanations or suggested using velocity to illustrate a solution. Two of the Asian students, Luis and Tim, were particularly adept about providing explanations involving velocity, rates of change, or graphs of functions.

During the post-teaching interviews, the participants developed a stronger understanding of the need for pedagogical content knowledge, especially KCT. Samantha understood this quite well.

> "I think this is why teaching for the first time takes a lot of time, you know? Because I know these things, but I never thought about explaining them to other people. So somehow, I don't even remember how I understood some of these things. So it takes a little bit of time to (find out) how much students actually know so you know what to base your explanations on, you know, what works with them."

All of the participants possessed strong common content knowledge. When presented with a mathematical problem such as the third problem from the post-teaching interview, most of them, but not all, could find several ways to solve a particular problem.

Question. Consider the following problem: Let

$$F(x) = \int_0^x \frac{1}{1+t^2}\,dt + \int_0^{1/x} \frac{1}{1+t^2}\,dt,$$

where $x \neq 0$.

(a) Show that $F(x)$ is constant on $(-\infty, 0)$ and constant on $(0, \infty)$.

(b) Evaluate the constant value(s) of $F(x)$.

How might students solve this problem? What difficulties would they encounter?

There are several solutions to this problem. One is found by resolving the indefinite integrals and writing $F(x) = \arctan(x) + \arctan(1/x)$, then using a trigonometric identity (the tangent of the complement of an angle is the reciprocal of the tangent of the original angle, as can be seen by drawing a right triangle) to realize this is expression identically equal to $\pi/2$ if $x > 0$ or $-\pi/2$ if x is negative. Another is found by changing variables in the second integral to form $\int_x^{\pm\infty} 1/(1+t^2)dt$, so that $F(x) = \int_0^{\pm\infty} 1/(1+t^2)\,dt = \pm\pi/2$.. A third solution (the one intended by the item author) is to apply the Fundamental Theorem of Calculus, combining it with the chain rule in the second integral, to show $F'(x) = 0$.

Most of the TFs found at least two ways to solve this problem, and several found all three of these, and all realized that this would be a very difficult problem for first-year calculus students. Many were quick to point out some of the difficulties that students might encounter. In the first solution, not all students will immediately remember that $\int_0^x 1/(1+t^2)\,dt = \arctan x$, and many students lack the fluency in trigonometry to simplify $\arctan x + \arctan(1/x)$. The second solution results in an improper integral, with which first-semester students may not be familiar. As for the third solution, students may not think to show $F(x)$ is constant by showing that $F'(x) = 0$, and combining the Fundamental Theorem of Calculus with the chain rule is a notorious source of difficulty. These examples illustrate how common content knowledge combined with some teaching experience lays the foundation for a pedagogical content knowledge base.

In contrast, we found that the pedagogical content knowledge of the TFs varied before their first teaching experience. In the third pre-interview question, the participants were presented with a situation where a student might have difficulty making sense out of a particular definite integral.

> Question. The graph of $f(x)$ is made up of straight lines and a semicircle. [See Figure 2.] We define the function $F(x)$ by

$$F(x) = \int_0^x f(t)\,dt$$

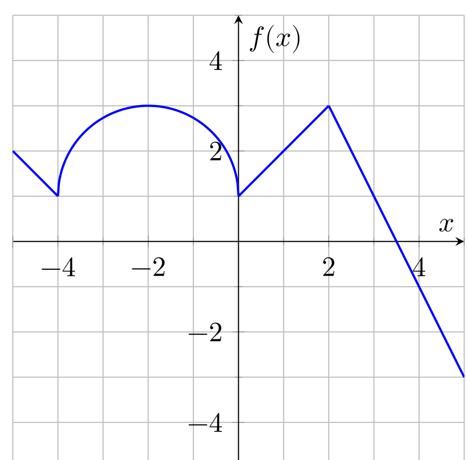

Figure 2. Graph for an interview question

> One of your students understands that $F(2) = 4$ but believes that $F(-2)$ is undefined. What would you say to the student?

Most of the TFs gave an explanation involving signed area, but only a few such as Amanda could come up with an explanation involving the definite integral as net change, "Well, the problem is, the students, they usually feel displacement is the distance walked. Distance is always positive for them." After giving an explanation using signed area, Christopher said,

> "You could say, why do we have this rule in the first

place? One reason for it is that we want the Fundamental Theorem of Calculus to hold. The Fundamental Theorem of Calculus says that differentiation and integration are related in the way we've learned about, that if we have the situation that $F'(x)= f(x)$, then the derivative of some point is equal to this function in here."

While this observation is certainly true, it is unlikely that first-year calculus students will possess the mathematical sophistication necessary to appreciate such an argument.

Limitations

Our investigation was qualitative in nature, and it is impossible to draw broad conclusions about the entire population of graduate students before and after their first teaching assignment. In addition, all of the participants were graduate students in a highly selective program. Therefore, one would expect that their common content knowledge would be very strong. However, as is the case with most graduate programs in mathematics, none of the participants in our study were admitted to the program on the basis of their potential teaching ability. The same is true of the students at this university. These undergraduates are on the whole highly motivated and exceptionally intelligent. However, those who took calculus-level mathematics courses exhibited the same behaviors and attitudes, including misconceptions and technical difficulties, that are typical of all undergraduate calculus students. The authors have taught at small and large private universities, state schools, and teacher's colleges, and they have found students at these diverse schools to be more alike than different when learning calculus.

Ideally, the graduate students in our study should have been interviewed before the pre-service teaching seminar. Interviewing graduate students at this time was not reasonable since many were new to the university and were occupied with qualifying exams, which are administered at the beginning of the fall semester.

Discussion and Conclusions

While pedagogical content knowledge develops with experience in the classroom, graduate students who have thought deeply about teaching before their first classroom experience understand that they must be able to follow student thinking to be a successful teacher. The following exchange with Samantha illustrates that even graduate students with strong CCK are not always aware of the source of their knowledge or how to employ it in their explanations to first-year college students.

A: "So yeah, no. This is one of the things that I— even when I did this apprenticeship, I realized that I had knowledge that came from somewhere at some point. But then I realized that I—Sometimes I'm using it without thinking where it comes from or how to explain it."

Q: "So the way you construct these things in your own head might be quite different from the way that your students construct them?"

A: "Well, yeah. First of all, some of these, I don't even question some of these things. I mean, I know that I proved them at some point and I know where they're coming from. I have an idea. But, you know, I'm not reproving everything in my head every time I'm using it. So yeah, it's a little— very different. And even when I—I don't know, even from the apprentice- ship part, it took me a while to read everything and figure out if that's the best way to explain it to people, especially after seeing how they interacted and what worked and what didn't work."

In conclusion, a strong pre-service teaching program should acquaint graduate students with the need for PCK in the classroom before their first teaching experience. Pedagogical content knowledge should be clearly defined during pre-service training and in depth examples should be provided. Furthermore, opportunities should be provided for graduate students to use their CCK to build PCK. Working with actual undergraduate students can provide opportunities to develop the ability to increase PCK once the TF enters the classroom as the primary instructor. Since PCK is acquired through classroom experience, a formal or informal mechanism for teacher training should continue through the remainder of the TF's career in graduate school.

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