In-Service Secondary Mathematics Teachers' Conceptions of Tangent Lines in Calculus

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Conflict of Interest Statement

The authors of this paper do not have financial or personal relationships with other people or organizations that could inappropriately influence or bias the content of this paper.

Abstract

Previous investigations into first year calculus students' understandings of tangent lines have revealed common misconceptions arising either from students' prior experiences with the topic or from the treatment of the subject in the calculus classroom. This study seeks to examine in-service secondary mathematics teachers' conceptions of tangent lines to see if similar misconceptions are held by this group. To this end, we conducted one-on-one interviews with 16 secondary-certified mathematics educators in which they were asked to complete an assessment which had them define, identify, and construct tangent lines. After analyzing the results, we found that the majority of the teachers that were interviewed did hold misconceptions that were similar to those misconceptions that are commonly held by first year calculus students.

Keywords: Mathematics Education, Tangent Lines, Teacher Education

1. Introduction

This study intends to analyze in-service secondary mathematics teachers' understandings of the concept of a tangent line and to relate these understandings to common student misconceptions of the topic. It serves as a follow up to a previous investigation of first year calculus students' misconceptions about tangency (Hogue and Scarcelli, 2021). In this former study, it was discovered that first year calculus students held misconceptions about tangent lines that are consistent with the teaching of tangent lines in geometry and algebra despite the fact that they had never learned about tangent lines, or could not remember learning about them, in either of those courses. Examples of these types of misconceptions include stating that a tangent line must only touch a curve at a single point or that a tangent line cannot "cross over" a function. This phenomenon not only suggests that students may acquire these kinds of misconceptions as a result of their education about tangent lines in geometry or algebra, but also that these misconceptions may arise entirely within a calculus classroom. Due to the variation and manifestation of these misconceptions, it is imperative to come to a better understanding of in-service mathematics teachers' knowledge about tangent lines in order to arrive at a better understanding of how these misconceptions originate. Our study seeks to investigate the following questions:

1. What, if any, misconceptions about tangent lines do secondary certified mathematics teachers hold?
2. If in-service secondary mathematics teachers possess misconceptions about tangent lines, are these misconceptions similar to those commonly found among undergraduates?
3. Do secondary certified mathematics teachers express a sufficient knowledge of tangent lines to teach this topic in a calculus setting?

2. Theoretical Perspective

In an analysis of mathematical misconceptions, Tall and Vinner's theory of the concept image (Tall and Vinner, 1981) provides a framework with which we can come to an understanding of how a students' own ideas about a concept interacts with the mathematical definition of the concept. Tall and Vinner define the concept image as "the total cognitive structure that is associated with the concept, which includes all mental pictures and associated properties and processes" (Tall and Vinner, 1981, p. 2). As the student encounters examples of a concept, works through problems involving the concept, is introduced to a formal or informal definition of the concept, etc., this concept image naturally changes to incorporate this new information. The concept image goes far be- yond a mere definition, which Tall and Vinner define as a "form of words used to specify the concept," and the concept image may or may not contain the generally agreed upon formal mathematical definition of the concept.

A student's concept image may also contain a personal concept definition, a definition that is constructed by the student using aspects of that student's concept image. This personal concept definition may contain aspects of the formal mathematical definition of the concept, or it may be entirely constructed by the student by referencing examples of the concept, informal definitions, personal experiences, etc. A student's personal concept definition need not be entirely in line with their concept image. For example, a student may know that the definition of a rectangle is a quadrilateral with all its interior angles being right angles and they may adopt this as their own personal concept definition of a rectangle, but this personal concept definition can be at odds with their concept image of rectangles if, for example, the student does not consider a square to be a rectangle.

If two portions of a student's concept image contradict each other, or if a portion of a student's concept image or concept definition is at odds with the formal definition of the concept, Tall and Vinner call this a potential conflict factor (Tall and Vinner, 1981). In the former situation, the student's ideas about the concept are self-contradictory, and they may very well discover and work through this contradiction without outside intervention, or through the mediation of visual or other examples of the concept. To give a simple example of this, a student's concept image about rectangles may contain the definition that a rectangle is a quadrilateral with four right angles. Yet, despite this, the student may not think that a square is a rectangle, simply because they have never seen a square referred to as such. In this situation, the student has all the knowledge necessary to work through this contradiction on their own, and we can see that it would be relatively easy for a teacher to correct this misunderstanding. The student knows the definition of a rectangle, they have simply never applied it to the special case of a square. The latter situation (a portion of a student's concept image is at odds with the formal definition of the concept) is more difficult. The student's concept image may be internally consistent, with contradictions only arising when this internal system is placed in opposition to the generally agreed upon mathematical theories. In such a situation, the student is usually unable to move past their misconceptions on their own, as from their own point of view their ideas contain no contradictions. Such misconceptions can only be overcome through the intervention of some outside influence, such as a teacher, textbook, fellow classmate, and so on.

The later situation is quite common among students who have misconceptions about tangent lines. For example, if we consider a student who believes that a tangent line is a line which touches a function at a single point with- out crossing over the function, this student can very easily apply this definition of tangency to mathematical examples without ever contradicting themselves. They simply accept as tangent lines those lines that fit this definition and reject those lines which do not fit. Proceeding in this way, it is impossible for them to contradict themselves at any point; contradictions can only arise if the student engages with the actual mathematical theory of tangent lines. However, engagement with the formal theory alone may not be enough to make the student aware of any contradictions. As Tall and Vinner note:

> Such factors can seriously impede the learning of a formal theory, for they cannot become actual cognitive conflict factors unless the formal concept definition develops a concept image which can then yield a cognitive conflict. Students having such a potential conflict factor in their concept image may be secure in their own interpretations of the notions concerned and simply regard the formal theory as inoperative and superfluous (Tall and Vinner, 1981, p. 154).

In other words, if a student comes to the realization that their own theories yield different and contradictory results to the formal mathematical theories, they may simply choose to ignore these formal theories. A tendency to ignore formal mathematical definitions has been noted previously among first year analysis and abstract algebra students at university (Edward, 1997; Edward and Ward, 2004) (for further discussion, see Section 3.2). If such a situation arises, it is unlikely that any contradictions will be resolved without the intervention of a teacher or some other authority figure.

Even if a student regards the formal theory with due diligence, they cannot become aware of the contradictions within their concept image unless their exposure to the formal theory brings these contradictions to light. One reason that misconceptions about tangent lines are so common among first year calculus students may be that their experience with the formal theory does not properly address these misconceptions, as will be discussed below (see Section 3.2).

3. Literature Review

The extant literature around tangent lines reveal seminal works and findings that address the mathematical significance and complexity around the teaching of tangent lines. The literature review will focus the challenges of teaching tangent lines, examples and misconceptions that provide a meaningful lens for research, and the mathematical content knowledge that undergirds this significant and often poorly-represented topic.

3.1 The Challenges of Teaching Tangent Lines

In geometry, tangent lines are typically introduced to students in reference to circles, where they are usually defined as lines which touch the circle at exactly one point. In the United States, an introduction to tangent lines in geometry is suggested by the Common Core Standards, which states that students be able to recognize that the radius of a circle is perpendicular to the tangent line passing through the endpoint of the radius (G-C2), and further suggest that students who intend to take advanced mathematics courses in the future (including calculus) learn to construct a tangent line to a circle that passes through some given point outside of the circle (G-C4) (National Governors Association Center for Best Practices & Council of Chief State School Officers., 2010). Students may also encounter tangent lines in algebra or pre-calculus, usually in reference to parabolas, but such a treatment is not prescribed by the Common Core standards and the topic is not commonly included in mainstream public textbook series (e.g., Larson and Boswell, 2018; Barnett, Ziegler, and Byleen, 2008). If tangent lines to parabolas are introduced in this setting, they are typically defined as lines which intersect the parabola at exactly one point without crossing over the parabola. Leikin and Winicki-Landman (2000, p. 20) discuss some common properties of tangent lines that a student may encounter in the classroom:

1. A tangent has only one point in common with a curve
2. There is a point that is at an equal distance from all tangent lines (as is the case with a circle)
3. All the curve is on one side of the tangent line (for functions continuous on the real numbers, this is equivalent to the statement that a tangent line does not "cross over" the curve)
4. The slope of the tangent line - if it exists - is equal to the value of the derivative of the curve's equation at the point of tangency.
5. A tangent is the graph of the linear approximation of the curve's equation (if it has one) at the point of tangency.
6. A tangent is the limiting position of secant lines passing through the point of tangency.

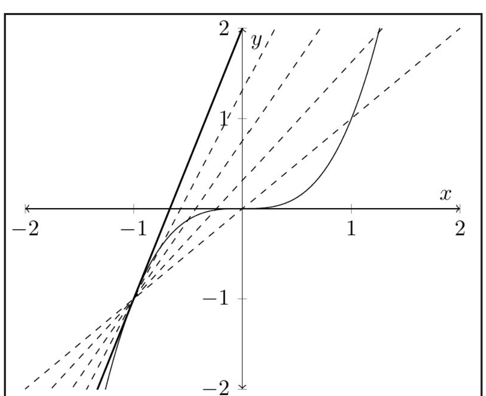

Figure 1: Approximation of a Tangent Line by Secant Lines

As they note, property 6 is the only property which gives a necessary and sufficient condition for a line to be tangent to a curve at some point (Leikan and Winicki-Landman, 2000). By "the limiting position of secant lines passing through the point of tangency," we mean that the secant lines passing through the points and "approach" the tangent line as $h \to 0$ (Assuming **f** has a tangent line at **x**, see Figure 1).

Despite the fact that the tangent line as a limiting position of secant lines (property 6) is the only property that provides a necessary and sufficient definition of a tangent line to a curve, student often incorrectly take some combination of the other properties as a personal concept definition of tangency (Biza and Zachariades, 2010; Biza, Christou, and Zachariades, 2008; Vincent, LaRue, Sealey, and Engelke, 2015; Vincent, 2016; Vincent and Sealy, 2015; Hogue and Scarcelli, 2021). For example, in a previous study (Hogue and Scarcelli, 2021) the following personal concept definitions of tangency were observed among first year calculus students in university:

1. Tangent lines "just touch" the graph of a function and do not cross over the function. (See Figure 2). This is an incorrect example of tangent lines drawn by a student in a prior study. Note that the lines drawn by the student in this figure were meant to pass through point a) (Hogue and Scarcelli, 2021)
2. Tangent lines are lines that touch the curve at only one point and are in direct correlation with the function's first derivative at that point.
3. Tangent lines intersect the function at only one point without crossing over the function and must be at a location on the graph where the derivative is 0.
4. Tangent lines are lines that touch and are parallel to the function but do not cross over it.
5. Tangent lines are lines that intersect the function without crossing over the function at the point of tangency.
6. Tangent lines represent *the rate of change of the slope* of a function, cannot cross over the function, and must intersect the function at only a single point.

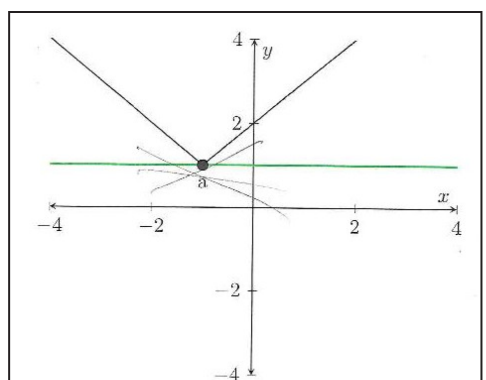

Figure 2. "Tangent Lines" Exhibiting Property 1

Here, we can see that students may apply Leikin and Winicki-Landman's properties 1, 3, and 5 in some combination to construct a personal concept definition of a tan-

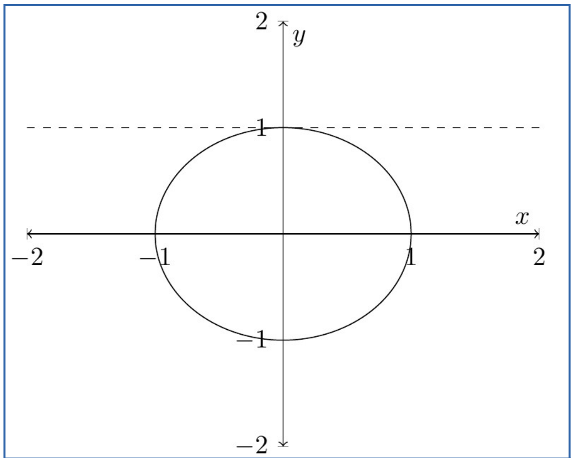

Figure 3. A Tangent Line to a Circle

gent line. We also see the use of some related properties, namely in the student who claimed tangent lines only exist at points where the derivative is 0 and the student who

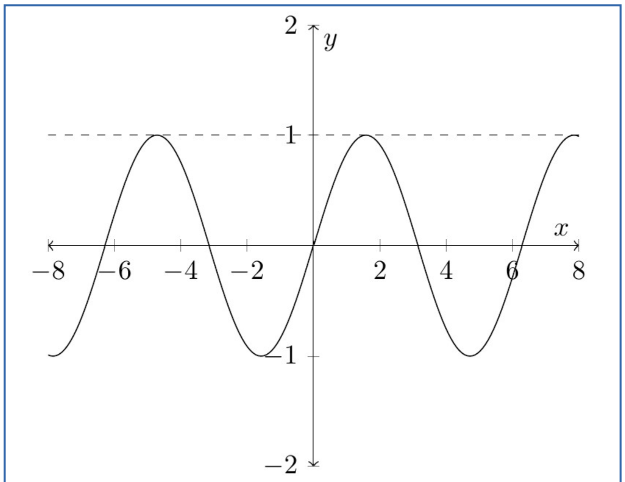

Figure 4. A Tangent Line Which Touches the Curve at Multiple Points, Without "Crossing Over"

claimed that tangent lines cannot cross over the function at the point of tangency. This later claim can be thought of as a "localized" version of property 3 (all the curve is on one side of the tangent line) as if we are working with continuous functions (which we typically are in first year calculus), the statement that a tangent line does not cross over the curve at the point of tangency is equivalent to the statement that the curve lies on one side of the tangent line in some neighborhood around the point of tangency. The tendency to adopt "local" versions of Leikin and Winicki-Landman's properties is discussed by Biza, Christou, and Zachariades (2008). They note that, in calculus, tangent lines depend on the local behavior of a function, being defined as the limiting position of secant lines passing through some tangent point. Yet, prior to calculus, tangent lines are defined by global properties, for example, touching the curve at only a single point or lying entirely on one side of the function (See Figures 3 and 4).

3.2 Examples and Student Misconceptions

Watson and Mason note that "the examples learners produce arise from a small pool of ideas that simply appear in response to particular tasks in particular situations" (2005, p. ix). This pool is called the example space. As we noted earlier, students with misconceptions about tangent lines may very well have a concept image that is internally consistent, hence it is of great importance to expose students to a wide variety of examples of tangency so that these contradictions with the formal theory can manifest themselves within the student's concept image. However, in first year calculus, a large majority of tangent lines shown all exhibit similar properties.

There is a tendency to overuse certain examples, such as a tangent line to a circle or to a parabola, in calculus textbooks (Biza and Zachariades, 2010; Kajander and Lovric, 2009) (For a tangent line to a circle, see Figure 3; For a tangent line to a parabola, see Figure 5). These two examples, a tangent to a circle and to a parabola, can be considered "super" examples (Hershkowitz, 1987), that is, examples that are very, perhaps overly so, popular. For example, in the latest edition of *Calculus Early Transcendentals* (Clegg, Watson, and Stewart, 2021), there are a total number of thirteen unique visual examples on tangency given in the lesson portion of the sections with tangents as a main topic (Section 2.1: The Tangent Line and Velocity Problems and Section 2.7: Derivatives and Rates of Change). Of these, eleven are of tangent lines that touch the function at exactly one point without crossing over the function (Section 2.1: Figures 1a, 2, 3, 5, 6; Section 2.7: Figures 1, 3, 4, 7, 8, 9), and two are of tangent lines that touch a function at two points (Section 2.1: Figure 1b; Section 2.7: Figure 6), although for one of these examples (Section 2.7: Figure 6) this intersection point is not actually shown since the tangent line is not extended far enough. No examples are given where the tangent line crosses over the function at the point of tangency, where the tangent line coincides with part or all of the function, or where the line is tangent to the function at a point where the derivative does not exist. Additionally, four of these visual examples are explicitly parabolas (Section 2.1: Figures 2, 3, 6; Section 2.7: Figure 7) and three appear to be 'parabola like' since they are either concave up everywhere or concave down everywhere and have the same general shape as a parabola for the portion that is drawn (Section 2.1: Figures 5 Section 2.7: Figures 8 and 9). This means that, just like in the case of parabolas, a tangent line to these functions will touch the function at exactly one point without crossing over it. As one visual example given is of a tangent line to a circle (Section 2.1: Figure 1), this means that, out of 13 total visual examples given, only 5 are neither circles nor 'parabola like'.

Though all examples of a concept fit the definition of that concept, various examples can have certain particularities that are not common to all examples of said concept. Mason and Pimm (1984) have noted that students may focus on the particularities of these examples, rather than focusing on how these examples fit the formal mathematical definition of the concept. For this reason, they argue that, when possible, teachers should give examples that are as general as possible so that there are no particularities for students to focus on. However, this is not possible for the concept of tangent lines. Though we can certainly derive a general equation of a tangent line to a differentiable function f at some point (such an equation is given by it is entirely impossible to construct a general visual example of a tangent line. Once we draw our function that is supposed to have a tangent, all generality is lost. This problem can be overcome by providing a wide variety of examples of tangency with varying properties. If this is done, then we can limit the particularities that students might focus on. For example, if one example that we show has the tangent line that touches at exactly one point of the function and another example shows a tangent line that touches at multiple points (See Figures 3 and 4), it would be less likely for a student to reach the conclusion that *all* tangent lines touch a function at exactly one point.

Accordingly, there are two issues that arise if we use a limited number of examples of tangency in a calculus class. In the first place, students who have an erroneous conception of what a tangent line is from their exposure to tangent lines in prior mathematics courses may find that their concept image or definition of tangency remains internally consistent if these examples do not properly challenge these false conceptions. On the other hand, students who are still in the process of constructing a concept image or definition of tangency may focus on the various

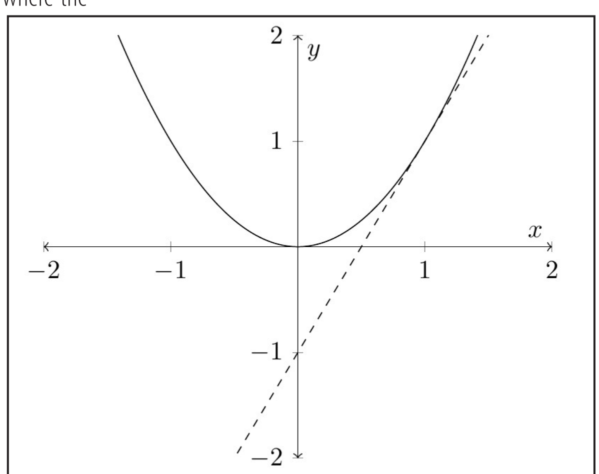

Figure 5. A Tangent Line to a Parabola

particularities of the tangent lines in this limited pool of examples and may mistakenly take these particularities as properties common to all tangent lines, that is, they may take these particularities as defining properties of tangency in their personal concept definition. For example, if a textbook uses only visual examples of tangents to circles or parabolas, students very well may very well conclude that all tangent lines share these properties. This may occur even if the student knows the formal definition of a tangent line in calculus. Vinner (1991) provided evidence that students often produce evidence from their concept image or concept definition rather than by consulting mathematical definitions, and this under-reliance on definition has also been shown by Edward and Ward (2004) among abstract algebra students and by Edward (1997) among analysis students. Additionally, this has been demonstrated by a previous study on the topic of tangent lines which found that students readily changed their personal concept definitions of tangency in response to various visual examples of tangent lines, even when they could not come up with a formal mathematical reason for doing so (Hogue and Scarcelli, 2021).

Everything that has been discussed up to this point places a huge emphasis on the teacher to come up with a variety of examples of tangent lines in order to help prevent misconceptions among the students. It may be the case that the teacher will have to develop these examples on their own, or at least without the assistance of a textbook, owing to the lack of a wide variety of examples in many textbooks that we noted previously, which means that it is imperative that the teacher has a solid understanding of tangent lines.

3.3 Mathematical Content Knowledge

When teaching any mathematical content, it is necessary to have a certain minimum understanding or prerequisite knowledge of that content in order to ensure that the information being delivered is accurate. It is of particular interest to our study to investigate whether or not secondary certified math teachers have that necessary knowledge needed in order to teach tangent lines. It is then a question of whether these teachers have sufficient mathematical knowledge for teaching (MKT) (Bass, 2005). Bass defines MKT as "the mathematical knowledge, skills, habits of mind, and sensibilities that are entailed by the actual work of teaching," and further divides MKT into four subcategories: (1) *common mathematical knowledge* (expected to be known by any well-educated adult), (2) *specialized mathematical knowledge* (mathematical knowledge that is unique to the teaching discipline); (3) *Knowledge of mathematics and students;* and (4) *Knowledge of mathematics and teaching* (Bass, 2005; Bass and Ball, 2004). In this work, we are principally concerned with the first subcategory; we want to know whether secondary teachers understand the topic on a basic, mathematical level.

It may be the case that secondary mathematics teachers in the United States are less prepared upon earning their certifications than teachers in other developed countries. In their investigation of middle school math teacher preparation, Schmidt, Burroughs, and Cogan (2013) establish "world-class standards" for mathematics teacher education programs by selecting courses taken by at least 80% of future teachers within at least 90% of the top performing teacher education programs (39 programs in total: one in Poland, 15 in Russia, 17 in Taiwan, and six in the United States). The set of nine core courses identified by this method included six university mathematics courses: beginning calculus, calculus, multivariate calculus, linear algebra, differential equations, and probability; two math education courses: math instruction and observing/analyzing math teaching; and one school mathematics course: functions/equations (Schmidt et al., 2013). They then say that teachers meet the world-class preparation bench- mark if they have taken at least 8 of these 9 courses. The results are quite bleak for US middle school preparation programs; only 31% of future teachers surveyed met this benchmark, while 95% or more met this benchmark in both Russia and Taiwan.

The positive correlation between a teacher's mathematical content knowledge and student achievement has been well documented (Clotfelter et al., 2010; Grouws and Schultz, 1996; Harbison and Hanushek, 1992; Tchoshanov, 2011; Thompson, 1992). Yet, many secondary teachers seem to be severely lacking in their content knowledge of calculus. Various studies have shown that a large number of secondary certified teachers lack a solid conceptual understanding of the subject (Huillet, 2005; Masteroides and Zachariades, 2004; Toh, 2009). Most studies limited themselves to examining teacher knowledge of differentiation, continuity, and integration; few studies have investigated in-service teachers' conceptions of tangency. Murrillo and Vivier (2013) showed that teachers can exhibit similar misconceptions as first year calculus students on the concept of tangency, but their investigation was limited in scale (five teachers) and included many questions that go beyond the knowledge taught in a first-year calculus course (for example, identifying tangent lines to parametric curves). Here, we intend to determine whether teachers will exhibit the same misconceptions when dealing primarily with standard curves appearing in most first-year calculus courses.

Other studies have shown misconceptions among certified teachers in other branches of mathematics. One such study, conducted with primary level pre-service teachers in Scotland, showed that a majority (80%) these prospective teachers were unable to accurately identify parallelograms from a group of quadrilaterals (Fujita and Jones, 2006), and another study, conducted with primary level pre-service teachers in Turkey, showed that only 51% of teachers interviewed were able to accurately identify parallelograms from a group of quadrilaterals and only 49% were able to accurately identify rhombi from a group of quadrilaterals (Erdogan and Dur, 2014). This lack of understanding of the differences between quadrilaterals, parallelograms, and rhombi among primary teachers is mirrored by a general lack of understanding among Turkish students at the secondary level (Aktas & Cansiz-Aktas, 2012; Biber et. al., 2019; Cansiz-Aktas, 2016), which suggests that this lack of understanding among teachers may be impacting student understanding in the long term.

4. Methods

We conducted our study with 16 secondary-certified mathematics educators who are currently teaching mathematics at the secondary level in either a public or charter school. A purposeful sample, as outlined by Patton (2002) and Suri (2011), was used to provide meaningful perspective and texture related to student understandings of tangent lines. We recorded the teachers' verbal and written responses to an assessment comprised of a series of questions on tangent lines in a face to face, semi-structured interview (Galleta, 2013) (see Appendix A). After these interviews, a thematic analysis was conducted on the teachers' responses to identify common patterns among the misconceptions exhibited by the teachers (Busi and Jacobbe, 2014). In total, five main themes were identified, which are discussed in section 5.3.

The participating teachers earned degrees and attended teacher training pro- grams at a wide variety of public and private institutions in the United States. All teachers who took part in this interview received their secondary teaching certification from the same state and had all

Question	Property 1	Property 3	Property 4	Property 5	Property 6
2.1	√				
2.2	√		√	√	√
2.3	√	√			
2.4	√				
2.5		√	√	√	√
2.6		√	√	√	√
3.1		√	√	√	√
3.2	√				√
3.3			√	√	√

Table 1. Demonstration of Leikin and Winicki-Landman's properties

	Mathematics Education Undergraduate	Mathematics Undergraduate	Unrelated Undergraduate	Master of Education
Teacher A	√			√
Teacher B			√	√
Teacher C	√			√
Teacher D	√			In progress
Teacher E	√			√
Teacher F	√			√
Teacher G	√			√
Teacher H			√	√
Teacher I		√		√
Teacher J		√		√
Teacher K	√	√		
Teacher L			√	√
Teacher M	√			√
Teacher N	√			
Teacher O	√			√
Teacher P		√		√

Table 2. Educational Backgrounds of Participants

been teaching in that state at the time that the interviews were conducted.

The assessment was designed to draw out any misconceptions the teachers might have about tangent lines, particularly those misconceptions that are related to Leikin and Winicki-Landman's properties. The first question used to allow teachers to demonstrate their current understanding of tangent lines (before encountering the graphical questions). This provided a baseline for comparison to see whether the ideas held by the teachers were changing throughout the interview. This introductory question was followed by a series of graphical questions, and finally three concluding written response questions to finalize the teachers' ideas about tangent lines.

The graphical questions in the assessment primarily focused on uncovering teacher misconceptions arising from Leikin and Winicki-Landman's properties 1 ("a tangent has only one point in common with the curve"), 3 ("all of the curve is on one side of the tangent line"), and 4 ("the slope of the tangent line is equal to the derivative of the curve's equation at the point of tangency") (Leikin and Winicki-Landman, 2000, p. 20). Questions 2.1 to 2.6 asked teachers to state whether a given line is a tangent line, questions 3.1 to 3.3 asked teachers to draw a tangent line to a given point on a function, and question 4 asked teachers to state whether a given line is tangent. Table 1 below outlines which of Leikin and Winicki-Landman's properties are exhibited by the lines given in each graphical question (see Figure 6) (Hogue and Scarcelli, 2021). For questions 3.1-3.3, the students were asked to draw the tangent line, so the properties given are for the correct tangent. As property 6 is the defining property of the tangent line, the given line is a tangent if and only if it fulfills property 6. Thus, this property is not checked for the incorrect tangents given in 2.1, 2.3, and 2.4. The graph given in question 6 has no tangent, so none of the boxes are checked. Property 2 has been excluded as it was not of much interest to our study.

	Pre-Algebra	Algebra I	Algebra II	Geometry	Trigonometry / Pre-Calculus	Calculus	Years of Teaching
Teacher A	√	○	○	√			14
Teacher B	√		√				24
Teacher C	√	√	√	√			17
Teacher D	√	√	√	√	√	√	7
Teacher E	√			√			11
Teacher F	√	√	√	√	√		21
Teacher G	√	√		√			16
Teacher H		√	√	○	○		15
Teacher I	√						2
Teacher J		√	√	√			2
Teacher K		√		√			3
Teacher L	√	√	○	○	○		8
Teacher M	√	√	√				4
Teacher N			√				15
Teacher O		○	○	√	√	√	29
Teacher P	√						2

Table 3. Subjects Previously Taught

4.1 Participants

The vast majority of participants (14/16) either had a masters degree in education, or, in the case of Teacher D, were currently earning such a degree. The most common educational background of the teachers interviewed was a Bachelor's degree in Mathematics Education with a further masters degree in education. Seven out of the sixteen teachers that were interviewed had such a background (A, C, E, F, G, M, and O) with an additional teacher (Teacher D) who had and undergraduate degree in mathematics education and was currently earning a master's degree in education. The educational backgrounds of the participants in our study are outlined below (see Table 2):

Three of the interviewed teachers (I, J, and P) had an undergraduate degree in mathematics with a master's in education and another three teachers (B, H, and L) had an unrelated undergraduate degree with a master's in education. Finally, one teacher, teacher K, had both a mathematics and mathematics education undergraduate degree, but no graduate degree, and another teacher, teacher N, had only an undergraduate degree in mathematics education. We note here that in the state where this study was conducted, teachers can obtain a secondary mathematics certification by passing the ETS Praxis II Mathematics Content Knowledge Exam, provided they already hold a teaching certification in some other discipline.

As previously discussed, tangent lines are dealt with in a variety of ways when studying different mathematical subjects at the secondary level. Thus, the subjects that teachers are most familiar with may have an impact on their understandings of tangent lines. The table below outlines the teaching experiences of the participating teachers (see Table 3). A checkmark (√) denotes that the teacher has taught this subject within the past five years. A circle (○) denotes that the teacher has taught this subject, but not within the past five years.

5. Findings

At the outset of each interview with our teachers, we sought to first determine their original conceptions of tangent lines with a few introductory questions. The tables below (Tables 4 and 5) gives a brief summary of the teachers' initial conceptions of tangent lines. These initial conceptions are based on the teachers' initial definitions of tangency as well as their responses to the first few graphical questions.

5.1 Teaching Experience and Initial Conceptions of Tangency

The majority of the teachers interviewed (Teachers A, B, C, E, F, G, H, K, L, M) had improper geometric or algebraic conceptions of tangency. Of these 10 teachers, only

	Touches at Exactly 1 point	Does not cross the function	Does not cross the function at the point of tangency	Some mention of derivate or slope
Teacher A	√	√		
Teacher B	√	√		
Teacher C	√	√		
Teacher D				√
Teacher E	√	√		
Teacher F	√	√		
Teacher G	√	√		
Teacher H		√		√
Teacher I				√
Teacher J				√
Teacher K	√			
Teacher L	√	√		√
Teacher M	√	√		
Teacher N				√
Teacher O				√
Teacher P				√

Table 4. Teachers' Initial Conceptions

two, Teachers H and L, noted any relation of the derivative or slope to the concept of the tangent line at the outset of the interview. Of the 8 teachers who did not note any relation between the derivative or slope and a tangent line, six noted that they had taught Geometry in the past 5 years (Teachers A, C, E, F, G, and K). Five of these six teachers, with teacher K as the exception, had taught the concept of tangent lines in their geometry class. Four other teachers who made no mention of derivative or "slope" noted that they had taught Algebra II in the past 5 years (Teachers B, C, F, and M) with one additional teacher, teacher A, having taught this class at some point and another noting that they had taught Trigonometry/Pre-Calculus in the past five years (Teacher F). None of the teachers who taught either Algebra II or Trigonometry/Pre-Calculus had taught about tangent lines in those courses.

The teachers' initial statements about tangent lines are summarized in Figure 10. Inexperienced teachers seemed more familiar with the definition of a tangent line that relies solely on the slope or derivative. Of the five teachers who had been teaching for five years or less (Teachers I, J, K, M, P), three of them mentioned derivative or slope in their initial definition of tangency (I, J, and P) and none of these three gave any geometric or algebraic conditions for tangency. For teachers who had been teaching for more than five years, five (Teachers D, H, L, N, and O) of the eleven mentioned derivative or slope in their definition, but only 3 of these (D, N, and O) did not give any geometric or algebraic conceptions of tangency. Two of these three teachers noted that they had taught calculus in the past five years, meaning that out of the 9 non-calculus teachers who had been teaching for more than 5 years, only one, teacher N, gave a definition of tangency that was entirely reliant on derivative or slope.

When tangent lines are considered in geometry, they are typically defined as lines which only intersect a circle at a single point. For our 9 teachers who had taught Geometry in the past 5 years, 6 had defined a tangent either as a line that intersects a function at a single point, or as a line which intersects a function at a single point without crossing over the function (these definitions are equivalent when considering tangent lines to circles). In addition, none of these six teachers noted any relation of the tangent line to the slope or derivative of a function. If we leave out the teachers who had also taught calculus, that leaves six out of seven geometry teachers who defined a tangent line in a way that is entirely in line with the concept of a tangent line to a circle. This lends some credence to the idea that students may develop improper ideas about tangent lines from their prior experience in other mathematics courses.

5.2 *Educational Experience and Initial Conceptions of Tangency*

It is important to discuss the various paths that a teacher in the United States can take in order to become a secondary certified mathematics teacher, as some readers may be confused as to why some of these teachers have a master's degree in education without an undergraduate degree in education, some have no math degree at all, and one (Teacher N) has no education degree. For the state in which all of the teachers in this study taught, which lies in the mid-Atlantic region of the United States, there are a few ways in which a teacher can become certified to teach a subject at the secondary level (grades 6-12). The most basic certification pathway is to earn ones certification in tandem with a bachelor's degree in education through an approved institution. Teachers who have a bachelor's degree in a related field can earn a certification to teach mathematics at the secondary level by earning a master's degree or a post-baccalaureate certification through an approved institution. It is worth noting that a master's degree program in education typically contains no subject area content, so that, usually, someone earning such a degree in curriculum and instruction will not have to take any mathematics courses as part of their degree. In addition, teachers who are already certified to teach a subject at the secondary level can earn additional certifications for other subjects at that level by passing the corresponding Praxis content knowledge test for that subject.

Those with a mathematics undergraduate degree generally gave better initial definition than those without such a degree. Of the four teachers (I, J, K, and P) with an undergraduate mathematics degree, three (I, J, and P) gave a definition that relied solely on slope or derivative. For those teachers without a bachelor's degree in mathematics, only Five out of twelve (D, H, L, N, and O) mentioned slope in their initial definition, with only three of these (D, N, and O) giving a definition that relied solely on slope. For the 9 teachers whose only undergraduate degree was in mathematics education (Teachers A, C, D, E, F, G, M, N, and O), 6 gave no mention of slope or derivative in their initial definition (A, C, E, F, G, and M).

5.3 *Misconceptions Demonstrated in the Interview Process*

The teachers that we interviewed exhibited a wide variety of misconceptions on the topic of tangency, though there were a few common themes which repeatedly arose throughout the interview process. Several of these themes were related to Leikin and Winicki-Landman's properties which were mentioned earlier, such as teachers believing that a tangent line only has one point in common with the curve, thinking that a tangent line cannot "cross over" a function, or noting some connection between the tangent line and the derivative. These erroneous conceptions lead some teachers to believe that a function could have multiple tangent lines at a single point, which is discussed in more detail below. Finally, another common theme which arose among the teachers was the idea that a linear function should not have a tangent line, whether their definitions allowed for this or not.

5.3.1 *A Tangent Line Has Only One Point in Common with a Curve*

At the outset of the interviews, nine teachers (teachers A, B, C, E, F, G, K, L, and M) had believed that a tangent line should only intersect a function at a single point. Yet, the teachers' applications of this principle varied, with some permitting exceptions to this rule for certain functions (namely, trigonometric functions). Additionally, throughout the course of the interviews, some teachers (A, C, and E) would drop this property as a necessary condition for tangency while others (E, G, and M) adopted a local version of this property for their personal concept definitions

Teacher A	The teacher was unsure how to define a tangent line at the outset of the interview, but did give the condition that a tangent line should intersect with "an arc of a circle." The teacher also believed that there was some relationship between the tangent line and the trigonometric function *tan(x)*, but could not say what the relationship was.
Teacher B	The teacher opined that "Tangent lines are lines right along the circle that are touching the outer circumference." The teacher also stated that they thought that tangents should only touch a function once, and without crossing over the function.
Teacher C	The teacher maintained that a tangent line is "perpendicular to a curve, hits the curve at one point [sic]." This 'perpendicular' condition was abandoned almost immediately. Additionally, they stated that the tangent line should only touch the curve once without crossing over the curve.
Teacher D	The teacher began the interview by giving the formal definition of a derivate, and stating that the slope of the tangent line should be equal to the derivative of the function at the point of tangency.
Teacher E	The teacher argued that tangent lines to any "shapes" would have to lie on the outside of that shape and could only touch the shape at a single point.
Teacher F	The teacher provided a singular response of "A tangent line will not pass though and only touch once (for any given function)."
Teacher G	The teacher held that a tangent line is "a line that just comes in and kisses an object. It only touches the object at one point." Through their response to the first two graphical questions (2.1 and 2.2), they also revealed that they did not allow a tangent line to "cross over" a function.
Teacher H	The teacher held that a tangent line is a line that is "extremely and infinitely close" to the function that has a slope that "matches" the slope of the function. They believed that a tangent line never actually touches the function, but instead occupies a position that is "infinitely close" to the function. As a result, they did not allow tangent lines to cross over the function that they are tangent to.
Teacher I	This teacher initially defined the tangent line that represents the "slope" of a function at the "connecting point." On question 2.3, they revealed that by "slope" of a function, they are referring to the derivative of the function at a point.
Teacher J	This teacher maintained that a tangent line is the "rate of change at which a function is changing at a given point."
Teacher K	This teacher initially defined a tangent line as a line that intersects the function at only a single point. However, on the first graphical question (2.1), they rejected the given line as a tangent line despite the fact that it met this condition. Their reasoning was that they did not like how the given line looked, and thy proceeded to draw the 'correct' tangent line. They stated that this correct tangent line was reproduced from their memory.
Teacher L	This teacher began the protocol by stating that a tangent line is a line that only touches the function at a single point without going through the function. The participant further stated that the derivate of the function at that point should be equal to the slope of the tangent line.
Teacher M	The teacher initially defined a tangent line as a line which touches the function at a single point without crossing over the function.
Teacher N	The teacher initially stated that "Tangent lines tell the history of smooth curve functions. Tangent lines reveal slope. Changes in slope predict future results." They also stated that tangent lines are related to first and second derivatives, but could not say what this relationship was.
Teacher O	This teacher began by stating that the tangent line is a line with a slope equal to the derivative of the function at the tangent point and proceeded to provide the correct limit definition of the derivative.
Teacher P	The teacher initially stated that the tangent line represents the "action" of a function (i.e. whether it is increasing, decreasing, or constant). They explained that a tangent line should "follow" the function.

Table 5: Teachers' Initial Definitions and Observations

of tangency. Only two of the teachers, teachers B and L, held quite firmly to the principle that tangent lines should only inter- sect a function at a single point, neither eliminating this property from their personal concept definitions of tangency during the interviews nor allowing for any exceptions to this rule. Teacher F, on the other hand, kept this property in their personal concept definition throughout the interview but did allow for one key exception. We will now examine some of the changes related to this property that the teachers made to their personal concept definitions during the course of the interviews.

The majority of teachers who changed their personal concept definitions with respect to this property in some way did so in response to a tangent line given at a minimal point on a sine curve (question 2.5). However, those teachers who did change their definitions in response to this question still did not permit tangent lines to intersect at more than one point without condition. The closest was teacher A, who changed their definition after encountering question 2.5 by stating that tangent lines could intersect a function more than once. Despite this, teacher A still rejected a tangent line to a linear function (which coincides with the function; question 2.6) since the tangent would intersect the function infinitely many times. When we pointed out that this would also be true for the tangent at a minimal point of a sine curve, they still believed that this would be a tangent but continued to reject a tangent line to a linear function.

After encountering question 2.5, teachers E, G, and M decided that a function could intersect a function at multiple points provided that the tangent line only intersect the function at one point locally around the point of tangency. They worded this property in various ways, with E and M stating that the domain should be restricted so that the tangent line only intersects at a single point (see Figure 6), and teacher G explicitly using the word "locally" in their statement of the property, but all gave equivalent characterizations. In Figure 11, the vertical lines drawn by the teacher represent the restriction of the domain, within which the tangent line only intersects the function at a single point.

Teacher F, on the other hand, continued to reject that a tangent line could intersect a function at multiple points despite the fact that they accepted the tangent line to a minimal point of a sine curve. For this teacher, sine curves were a special type of curve for which the general rules of tangency did not apply. Thus, a tangent line to a sine curve could intersect that curve multiple times, but this was not true for other functions.

Teacher C had initially stated that a tangent line could only intersect the function at one point, but eventually dropped this condition when they adopted the condition that a tangent line should be perpendicular to the x or y axis. It is unclear what prompted this change, the teacher simply stated that they had remembered that this was the case.

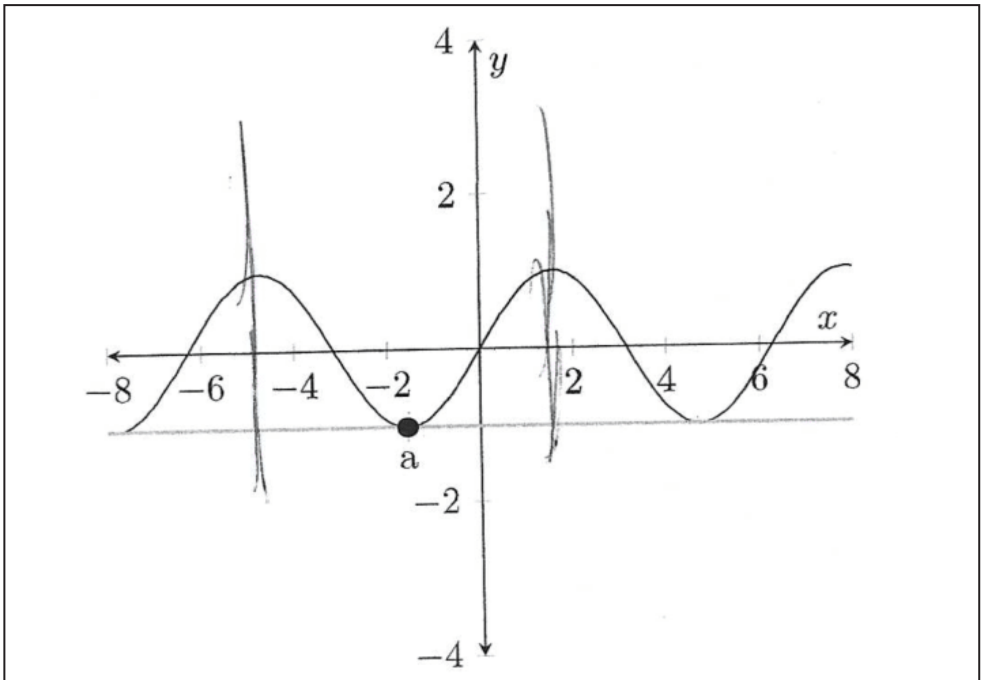

Figure 6. Restriction of Domain by Teacher E

5.3.2 A Tangent Line Does Not "Cross Over" the Function

Initially, nine teachers (A, B, C, E, F, G, H, L, and M) had believed that a tangent line could not "cross over" the function that the line is tangent to. Seven teachers (A, B, E, F, G, L, and M) still held this believe at the end of the interviews, while teachers C and H eventually would drop this condition from their personal concept definitions.

For those seven teachers who held firm in their belief that a tangent line could not cross over a function, they applied this property consistently throughout the interview without contradicting themselves. Thus, any tangent line that crossed over a function was rejected, and when asked to draw tangent lines they never drew one that crossed over a function and stated that the tangent line did not exist if it was not possible to draw a line that would not cross over the function (for example, questions 3.2 and 3.3). One exception to this was teacher B's use of 'tangent rays'. In response to question 3.2, teacher B drew a 'tangent ray' and stated that this could be a tangent line since it did not cross over the curve (See Figure 7).

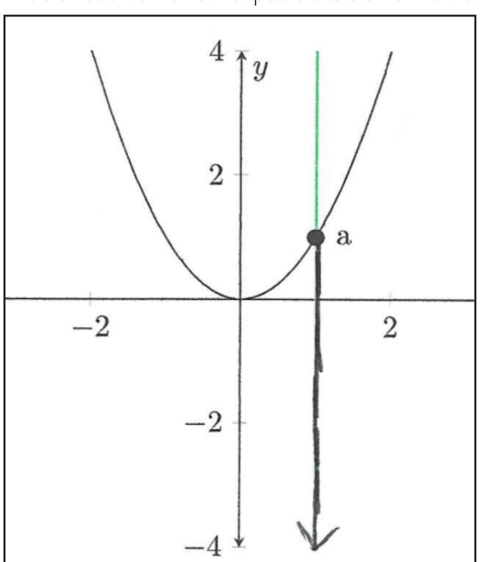

Figure 7. An Example of Teacher B's "Tangent Rays"

Teacher C, as mentioned previously, changed their personal concept definition during the course of the interview, deciding that tangent lines should be perpendicular to the *x* or *y* axis. In doing this, they dropped to prior conditions given, namely that tangent lines could only intersect a function once and that tangent lines could not cross over a function. After this change, they had no issues with accepting tangent lines that cross over a function.

Teacher H had initially stated that tangent lines should lie "infinitely close" to a function without touching the function. Thus, a tangent line cannot cross over a function, as, in order to do that, it would have to touch the function. They would reconsider this position later on when encountering question 3.3 (where they were asked to draw a tangent line to a non-minimal or maximal point on a sine curve) and finally decided that a tangent line could actually cross a function. They had no specific reasoning for this change, but stated that some information about tangent lines was "coming back" to them.

5.3.3 Tangent Lines and Calculus Properties

This section will primarily deal with the teachers who related tangent lines in some way to the derivative, slope, or rates of change, but teachers who related tangent lines to other calculus properties are also included. In total half of the teachers (D, H, I, J, L, N, O, and P) made some relation between tangent lines and the derivative or slope of a function at both the start and end of the interview, while two teachers (E and K) did not initially include the derivative or slope of a function in their personal concept definitions but did add them in by the end of the interviews.

Despite the large number of teachers who mentioned slope or the derivative in their initial concept definitions, few did so entirely correctly. Only three teachers (D, J, and O) gave what could be considered as a correct definition of tangency for continuous functions (that the slope of the tangent line is equal to the derivative of a function at the point of tangency). These teachers responded correctly to all questions in the protocol, with the exception question 3.2, which asked the teachers to draw a tangent line at a point where the derivative of the function is undefined (in this case, the tangent should be vertical). Teacher D rejected a tangent line at this point, correctly stating that the derivative would be undefined. Teacher J noted that the derivative would not be defined at this point, but noted that the rate of change of the function would "approach infinity" at that point, leading them to conclude that the tangent should be vertical. Finally, teacher O noted that the secant lines of the function approach a vertical line at that point, hence the tangent line should be vertical. We should note that teacher O was the only teacher to apply the definition of the tangent line as the limiting position of secant lines at any point throughout the whole interview process.

Other teachers related the tangent line to the derivative, slope, or rate of change of a function with varying

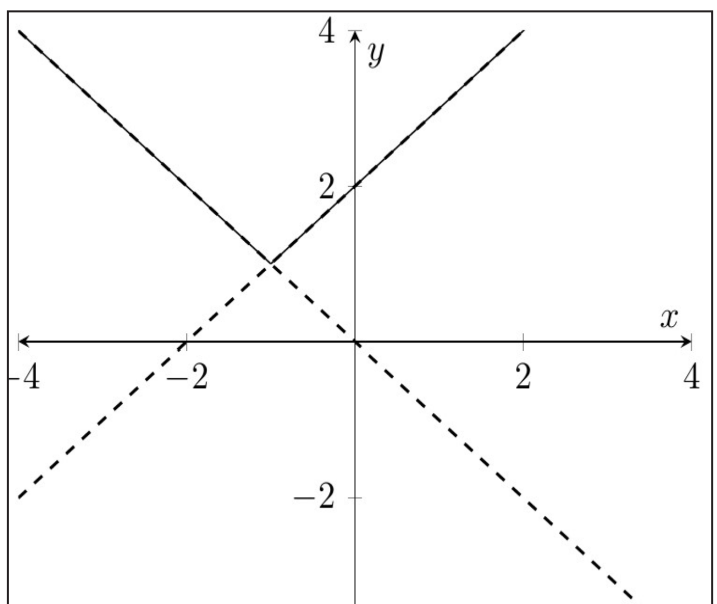

Figure 8. Teacher H's "Tangent Lines" to an Absolute Value Graph

degrees of success. Teacher E had initially made no connection between tangent lines and the derivative or slope, but added this to their definition in response to question 2.3 (tangent line to a minimal point on an absolute value function). They noted that the given line was not "going along with the function" and stated that the slope of the tangent line and the slope of the function should me "similar". In the end, they added to their final personal concept definition the statement that the "slope of the tangent line would match whether the function is increasing or decreasing [sic]."

Teacher H noted in both their initial and final definitions of tangency that the tangent line has a slope that "matches" the slope of a function. As noted previously, however, they also required that a tangent line lie "infinitely close" to a function, a definition that may be related to the limit process. Although they eventually altered this property to allow tangent lines to cross over functions, they did still apply it in response to question 4, which asks whether the vertical asymptote to $f(x) = \frac{1}{x}$ is a tangent line. They argued that, since this line lies "infinitely close" to the function, it should be a tangent line. Teacher H seemed to struggle with the tangent line to the edge point of an absolute value graph (2.3). They eventually decided that this function would have tangent lines at that point, but that they should be the continuation of the two linear components of the graph (See Figure 8).

Teacher I initially stated that the tangent line "represents the slope of a function" at the "connecting point". They revealed in response to question 2.3 (the tangent line to the minimal point of |x|) that by slope, they mean the derivative of the function. However, they incorrectly concluded that the derivative of the function f (x) = |x| at x = 0 is equal to 0, allowing them to incorrectly accept the given line as a tangent line. In their final personal concept definition of tangency, teacher I noted that the tangent line tells us the slope at the point of tangency.

Teacher K initially made no mention of the slope or the derivative of a function in their definition of tangency, but changed this definition in response to question 2.3. Teacher K stated that they were bothered by the "pointiness" of the function. It was here that the teacher recalled that the tangent line had something to do with the first derivative of the function. They argued that since the function would not have a first derivative at the intersection point, the tangent line should not exist. They further reasoned that a tangent line should tell us about the zeroes, or x-intercepts of a function. Because of this, they argued that the graph in question 2.1 (which gives a tangent line to a parabola) should have exactly two tangent lines, and that these tangent lines should tell us the zeroes of the original function. Teacher K was unable to reach a consensus about the nature of derivatives during the course of the interview. On some questions, Teacher K seemed to treat the derivative as a rate of change (for example, in response to question 3.2 they argued that the tangent line would have to be vertical, but stated that a vertical rate of change was not possible), yet on others they argued that the derivatives were actually just the zeroes of the original function. In their final definition, they stated that the tangent line is the first derivative.

Teacher L stated that the slope of the tangent line should be equal to the derivative of the function at both the start and end of the interview, but also added that a tangent line should only intersect a function once and should not cross over the function. This caused them to reject the tangent line in question 2.2 (which gives the correct tangent line to the function $f(x) = x^3$ at the origin) as even though they recognized that the derivative was 0 at that point, the tangent line would cross over the function.

Teacher N initially stated that "Tangent lines tell the history of smooth curve functions. Tangent lines reveal slope. Changes in slope predict future results." They also claimed that tangent lines had some relation to the derivative and that the second derivative also plays a role, but they were not able to say what the exact relation was. Their ideas about the topic were made more clear in their response to question 2.3 (graph of absolute value function) in which they correctly stated that the function would not have a tangent line at the given point as the slopes approaching from the left and right are not equal. Though Teacher N noted that there was some relationship between a tangent line and the derivative, they were unable to clearly state what this relationship is. They did note that one could find the formula of some tangent lines by using point- slope form, setting the slope equal to the derivative, but they also stated that this was not true in all cases. An example of this can be seen in their response to question 4, which asks the teachers whether the vertical asymptote to the graph of $f(x) = 1/x$ is a tangent line. They argued that, since this line lies "infinitely close" to the function, it should be a tangent line. The teacher responded that it is, since it "describes the split" of the function. They did, however, drop their claim that the tangent line is related to the second derivative, noting that the second derivative describes concavity.

Teacher P began by stating that the tangent line represents the "action" of a function (i.e. whether it is increasing, decreasing, or constant) and further stated that a tangent line should "follow" the function. When asked what they meant by "follow", they replied that the behavior of the tangent line should be "similar" to the behavior of the function at the point of tangency. Though they never referred to this "behavior" as a slope or a derivative, most of their responses seemed to be in line with the concept of a tangent line as a line that matches the derivative of a function. Some notable exceptions included question 2.3 (tangent to an absolute value graph) where Teacher P claimed that the given line was a "good representation of what is happening to the function at the given point" and question 2.6 (tangent to a liner function) where they claimed that a tangent line could not be the function itself. Teacher P defined tangent lines as "a representation of the behavior of a function at a given point" at the conclusion of the study.

Teacher C, who did not make any relation between the tangent line and the slope or derivative of the function at any point during the interview, did relate tangent lines to another calculus topic. As previously noted, teacher C required that tangent lines be perpendicular to the x or y axis, and further stated that a tangent line would represent the "area under the curve up to that point". They did not, however, include this property in their final definition of tangency, although they still believed it to be true at the conclusion of the interview. This would appear to be a confusion between the topics of differentiation and integration on the part of the teacher.

5.3.4 Can There Be Multiple Tangent Lines at a Single Point?

One question that was not directly tested by any of our graphical questions, but which was still of interest to our study, was the questions as to whether a single point on a function could have multiple tangent lines. We asked this question directly at the end of our protocol, after teachers had given their final definition of tangency. In total, four teachers (teachers A, C, G, and M) allowed for multiple tangents at a single point on a function. For the most part,

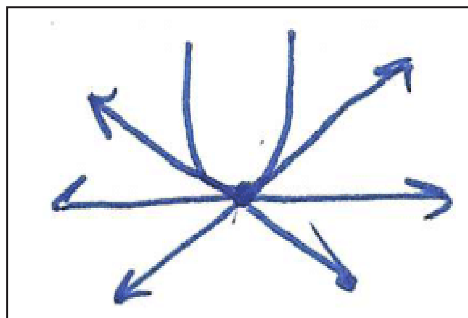

Figure 9: . Multiple Tangent Lines Drawn by Teacher M

the teachers' responses to these questions can be inferred from their final definitions of tangency, so that teachers who defined tangents in relation to the derivative/slope did not allow multiple tangents to a single point because of the uniqueness of the derivative/slope and teachers who did not mention the derivative or slope did allow for multiple tangent lines at a single point (See Figure 9). Notable exceptions are teachers B and F; neither mentioned derivative or slope in their definitions of tangency, but both did not allow for multiple tangent lines at a single point on a function.

Teacher B asserted that it was not possible to have multiple tangent lines at a single point as the tangent line would have to be "perpendicular" to the function. We noted previously that what Teacher B meant by "perpendicular" was not entirely clear as many of their responses included tangent lines that were not perpendicular to the given

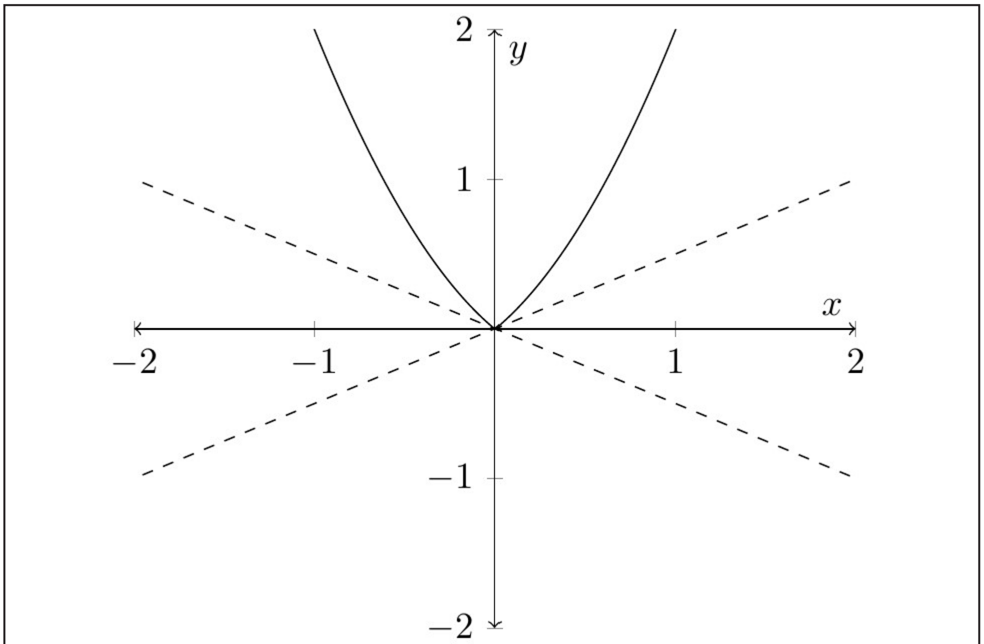

Figure 10: Two "Tangent Lines" at a Cusp Point

function. We asked Teacher B to clarify their meaning, but were still unable to determine what they meant by this. Teacher F defined a tangent line as a line that intersects a "curved" function at only one point without crossing over that function. Teacher F stated that by "curved," they meant a function that is non-linear, and they did not allow for tangent lines to functions that were not "curved." They stated that for a point on a curved graph, there could be at most one line that intersects this point and no other point on the function without crossing over the function. Though, using Teacher F's definition of "curved," this statement is not correct (we could have a non-linear function that has a cusp point) (see Figure 10).

5.3.5 Tangent Lines to Linear Functions

As can be seen in some of the examples given above, there was a tendency among the teachers to reject tangent lines to linear functions. In total, 12 of the teachers that we interviewed (A, B, E, F, G, H, I, K, L, M, N, and P) did not allow for tangent lines to linear functions. For some teachers, this was perfectly in line with their definition of tangency, as was the case for teachers A, B, E, F, G, K, L, and M, but for others (H, I, N, and P) linear functions represented special cases which should be treated differently than other functions.

For Teacher H, a tangent line should be "infinitely close" to a function without touching the function (although they did make some exceptions to this rule) and should "match the slope" of the function. Initially, Teacher H rejected a tangent line to a linear function because they thought that any such line would have to touch the function, which they did not permit in general. However, they began to consider that it might be possible to place a line similar to the original function infinitely close to it, and that this line may be a tangent. They eventually rejected this, because they claimed that this line would give no new information about the original function. Teacher N made a similar claim that the tangent line would provide no additional information about the function, and thus should not exist. Teacher I, on the other hand, argued that if a tangent line to a linear function existed, then it would simply be the original linear function and would thus be tangent to the function at multiple points, which they did not permit. Finally, Teacher P rejected a tangent line to a linear function simply because such a line would have to be identical to the function. Teacher P stated that they believed that a tangent line could coincide with a portion of a function, but not with the entire function, although they did not have any reasoning for this claim, simply citing prior experience.

5.4 Summary of Findings

The following section shall detail summary findings from the study.

As Tables 6 and 7 show, many teachers in our study still held improper ideas about tangent lines at the conclusion of the interviews. Note that some of the properties mentioned by the teachers during the course of the interviews were not included in their final definitions, even if they still believed some of these properties to be true. Half of the teachers held improper geometric or algebraic conceptions of tangency, either stating that a tangent line could only touch a function at a single point or stating that a tangent line could not cross over the function. In addition, six out of the sixteen teachers made no reference to the derivative in their final definitions of tangency. However, there was still a general improvement in the conceptions of our teachers, with the total number of teachers with improper geometric or algebraic conceptions of tangency being reduced by two and with the total number of teachers mentioning derivative or slope in their definition increasing by two. Still, only seven teachers gave a definition of tangency that was solely reliant on derivative or slope, and even among these teachers some misconceptions were observed. For example, Teacher N did not accept tangents to straight lines and considered some asymptotes as tangent lines. Only three teachers, Teacher D, Teacher J, and Teacher O, correctly applied the calculus definition of a tangent line to a differentiable function throughout the entire protocol without exhibiting any

	Touches at Exactly 1 point	Does not cross the function	Does not cross the function at the point of tangency	Some mention of derivate or slope
Teacher A		√		
Teacher B	√	√		
Teacher C				
Teacher D				√
Teacher E		√		√
Teacher F	√	√		
Teacher G	√	√		
Teacher H				√
Teacher I				√
Teacher J				√
Teacher K	√			√
Teacher L	√	√		√
Teacher M	√	√		
Teacher N				√
Teacher O				√
Teacher P				√

Table 6: Teachers' Final Conceptions

Teacher A	This teacher held that "a tangent line is a straight line that intersects a curve or an absolute value function at a specific point without moving from one side of the graph to the other." The teacher made an exception for the example for the example of a tangent to a sine curve, saying that whether a tangent line could touch a function at more than one point depended on the graph being considered.
Teacher B	The teacher maintained that a tangent line is a line which touches the function only once without crossing over and is perpendicular to the function.
Teacher C	This teacher opined that tangent lines are lines passing through a function that are perpendicular to either the x or y axis and represent the area under the curve up to that point.
Teacher D	The teacher noted that the slope of the tangent line should be equal to the derivative of the function at the point of tangency. This perspective remained unchanged from the initial conception.
Teacher E	The teacher stated "A tangent line is a line that passes through a point on the outside of the figure. The slope of the tangent line would match whether the function is increasing or decreasing."
Teacher F	This teacher contended "A tangent line to a function is a line that intersects a curved (meaning non-linear) graph at one point but would not cross over the graph."
Teacher G	The teacher held that a tangent line is a "linear equation that intersects the function at a particular point without crossing over."
Teacher H	This teacher maintained a tangent line is "a line that is infinity close to a curve and whose slope matches the slope of the function."
Teacher I	The teacher defined tangent line as a "straight line that is unique to a point on a graph that tells us the slope at that point." They clarified that by "unique to a point," they meant that a tangent line could only be tangent to a single point on a function. They also stated that the tangent line could touch the function multiple times."
Teacher J	The teacher stated that a tangent line is the "rate of change at which a function is changing at a given point." This perspective remained unchanged from the initial conception.
Teacher K	This teacher proposed that a tangent line is the "first derivative, cannot be a vertical line, must touch the graph at only one point, in some cases, tangent can exits if given restrictions (e.g. sine function). If the function has no derivative, it has no tangent [sic]."
Teacher L	The teacher upheld that a tangent line is a line that only touches the function at a single point without going through the function and further stated that the derivative of the function at that point should be equal to the slope of the tangent line. This perspective remained unchanged from the initial conception.
Teacher M	This teacher defined a tangent line as a line which touches the function at a single point without crossing over the function. They stated that tangent lines to "non-curved" functions do not exist. This perspective remained unchanged from the initial conception with the exception of stating that non-curved functions should not have tangent lines.
Teacher N	The teacher held that "Tangent lines tell the history of smooth curve functions. Tangent lines reveal slope. Changes in slope predict future results." This perspective remained unchanged from the initial conception apart from the removal of any relationship between the tangent lines and the second derivative.
Teacher O	This teacher said that the tangent line is a line with a slope equal to the derivative of the function. This perspective remained unchanged from the initial conception.
Teacher P	The teacher contended that tangent lines are "a representation of the behavior of a function at a given time."

Table 7. Teachers' Final Definitions and Observations.

6. Conclusions and Discussion

Our study reveals that many secondary certified mathematics teachers' conceptions of tangency contain similar misconceptions about the topic to those that are commonly found among first year calculus students (Hogue and Scarcelli, 2021). The vast majority of teachers that we interviewed held improper ideas about tangent lines and gave personal concept definitions that were not in line with the formal definition of tangency, with only three out of sixteen teachers (teachers D, J, and O) responding to all questions without exhibiting any major misconceptions. This has some serious implications for the preparation of mathematics teachers. Though it is true that the teachers who had taught, or are currently teaching calculus in our study (Teachers D and O) performed well in our study, serious problems could arise if any of the thirteen teachers who did exhibit misconceptions should be required to teach this course at some point in the future. It is, of course, not at all unlikely that such a teacher will end up correcting their own misconceptions through their preparation to teach the class, yet the problems with the treatment of this topic in textbooks mentioned earlier could cause some of these misconceptions to persist.

We are not attempting here to put forward an indictment of America's secondary mathematics teachers; obviously there will be certain topics that fade from a teacher's memory over the years. Rather, we only intend to point out yet another difficulty which may arise in teaching calculus on the secondary level: that many teachers simply are not comfortable with the content. Calculus is a difficult subject, and one would not expect someone to remember every single detail if that person does not regularly work with the subject.

One concern arising from these misconceptions among teachers is that, should a teacher with these misconceptions be asked to teach a calculus class, they may unintentionally transfer these misconceptions to the students. absolute value graph (question 2.3) and a tangent to a graph at a point where the derivative is undefined (question 3.2), so these examples might also prove useful in the classroom. Two teachers also incorrectly identified a vertical asymptote as a tangent line (question 4), hence it might also benefit students to provide this as a non-example of tangency.

Beyond the need to teach calculus, such misconceptions can also have underlying implications for fostering misconceptions with underclass students. If many secondary certified mathematics teachers do not have a proper under- standing of tangent lines in calculus, then this lends credence to the idea that some of these misconceptions have their origins in the students' prior mathematics experiences, especially considering the fact that six out of nine geometry teachers interviewed gave a definition of tangency that did not at all rely on the derivative of a function. It is possible, then, that these teachers might give students a conception of tangency that is correct in reference to circles, but not correct for future studies of tangent lines. Such a conception may stick with the students throughout their calculus studies.

We noted earlier that our more inexperienced teachers seemed more comfortable with defining a tangent line solely in terms of the derivative or slope. If more time spent as a teacher really does have a destructive effect on teachers understanding of tangency, then this is likely because the calculus definition has gone unused for so long that it has simply been forgotten. The fact that both calculus teachers in our study gave definitions of tangency that were based solely on the derivative despite the fact that they had been teaching for more than 5 years would then be explained by their continual exposure to the topic in a calculus setting. There is, however, an alternative explanation to the performance gap between more experienced and less experienced teachers in that these less experienced teachers were more likely to have an undergraduate degree in mathematics. In all likelihood, both factors contribute to the performance gap to some extent.

The problems that may arise if the teacher of a calculus class holds misconceptions about tangency are more obvious. There is a risk that these misconceptions may take root with the students, and, given the lack of variety in visual examples of tangency shown in many textbooks, the student may be unable to overcome these misconceptions on their own (and the same goes for the teacher). Given the large percentage of teachers in our study that exhibited misconceptions about tangent lines (13/16, or about 80%) it seems not at all unlikely that there are quite a few new or even experienced calculus teachers that hold similar misconceptions even though neither of the two calculus teachers in this study had any serious misconceptions.

Our study may suggest that in-service mathematics teachers harbor significant misconceptions related to their understanding and application of tangency- based principles. This suggests that while greater efforts are espoused for pre-service math teachers' engagement with mathematical content, the understanding of tangent lines may require additional emphasis in teacher preparation programs. Given the research that has been done on secondary teachers under- standing of other concepts of calculus it appears that this additional emphasis may need to encompass more than just tangent lines in calculus (Masteroides and Zachariades, 2004; Huillet, 2005; Toh, 2009).

What can be done to remedy this issue lies outside of the scope of this paper, but the implications for the secondary calculus classroom make it a serious problem worthy of further consideration. We can, however, offer a few suggestions to improve student understanding of tangent lines in the calculus classroom. Teachers should employ a wide variety of examples of tangency and avoid overused examples of tangent lines such as a tangent line to a circle or a parabola to ensure that students are exposed to tangent lines with a variety of properties.

Our interviews revealed a few specific examples of tangency that might be particularly useful toward this end. Many of the teachers that we interviewed who initially stated that a tangent line could only cross a function at a single point changed their personal concept definitions of tangency by either removing or altering this requirement in response to a tangent line at a minimal point of a sine curve (question 2.5). This suggests that such an example may be effective at correcting this misconception in calculus students. For some teachers, however, this example only managed to slightly improve their personal concept definitions, with the teachers instead adopting a local version of this property. Thus, it may also be beneficial to show an example of tangency where the tangent line coincided with the function in a region around the point of tangency such as a tangent line to a linear function, which was another example that caused trouble for a number of our teachers.

Other examples of tangency that our teachers struggled with include a tan- gent line to a minimal point on an absolute value graph (question 2.3) and a tangent to a graph at a point where the derivative is undefined (question 3.2), so these examples might also prove useful in the classroom. Two teachers also incorrectly identified a vertical asymptote as a tangent line (question 4), hence it might also benefit students to provide this as a non-example of tangency.

It is not enough for teachers to simply present these examples as tangent lines or non-tangent lines, they must also explain why these examples are examples of tangent lines by employing the definition of tangent lines so that students can learn how to use that definition to identify and construct tangent lines on their own. Finally, teachers should explain how the definition of tangency used in calculus differs from previous definitions that students might have learned so that students who have learned about tangent lines in prior mathematics courses can move past these old definitions. Though these suggestions will not improve in-service teachers of mathematics understandings of tangent lines, they do have the potential to improve future teachers of mathematics understandings of the subject if employed in pre-service teacher preparation programs.

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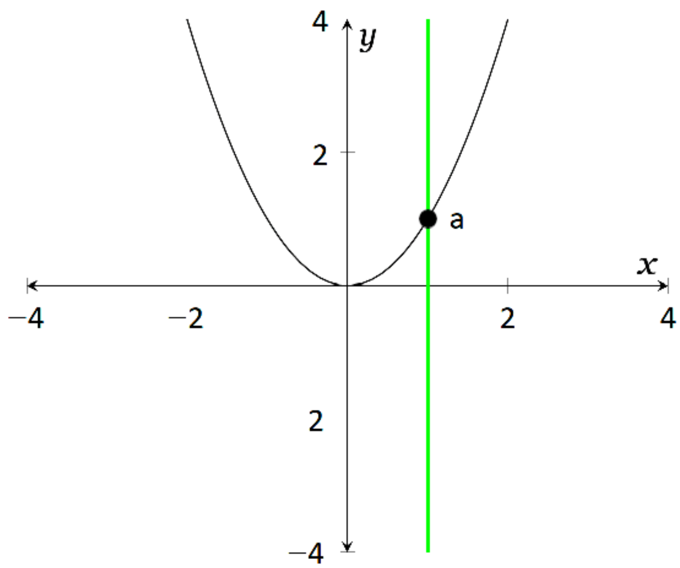
Appendix A

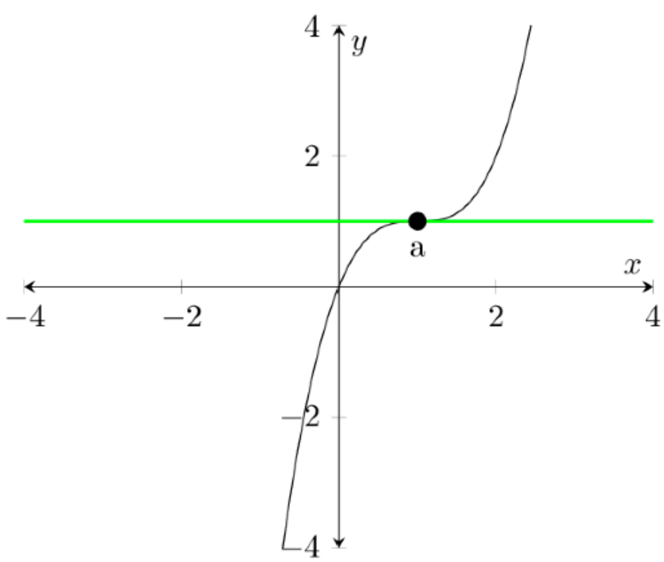
What do you know about tangent lines?

In which of the following graphs is the green line a tangent to the function at point a? Explain your reasoning for each graph.

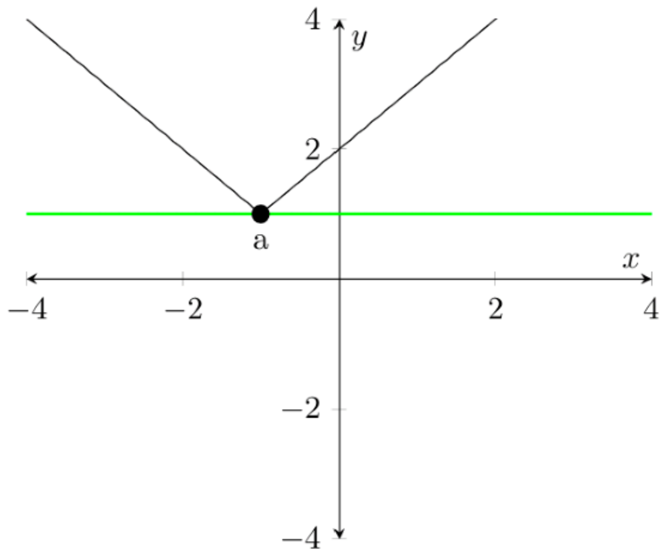
1



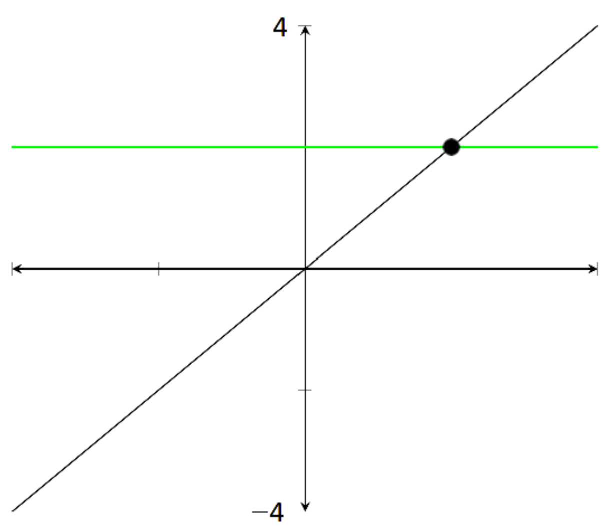
2



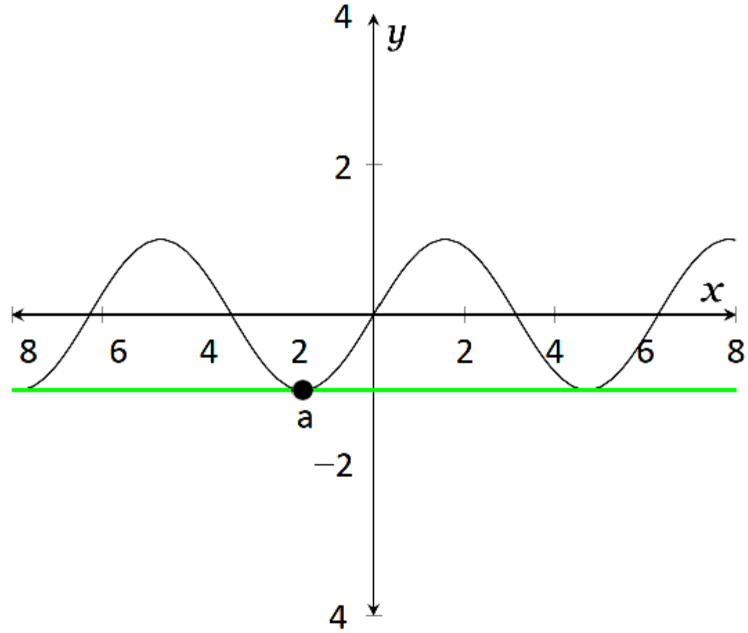
3



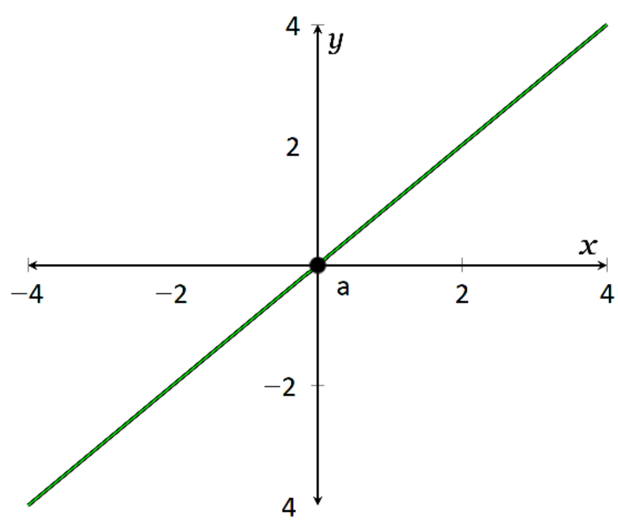
4



5

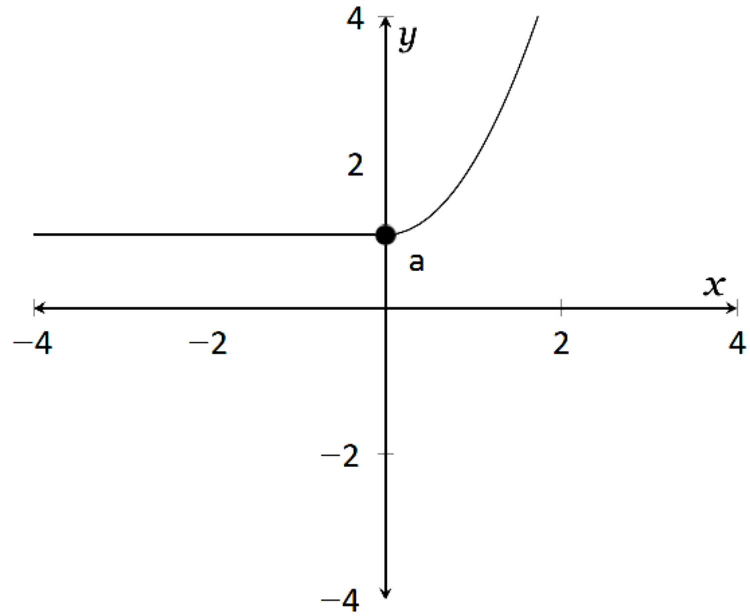


6

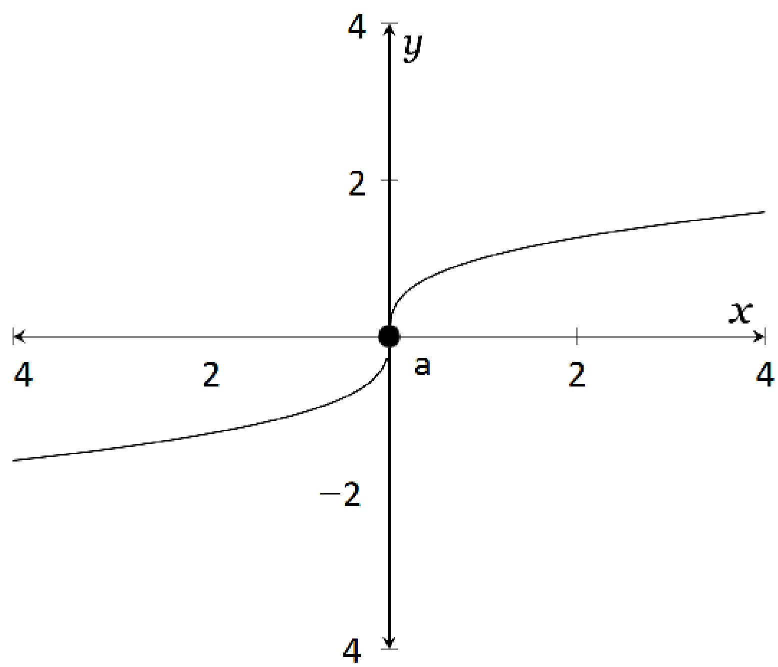


Draw a tangent line to the function at point a. Explain your answer (If it does not exist, explain why)

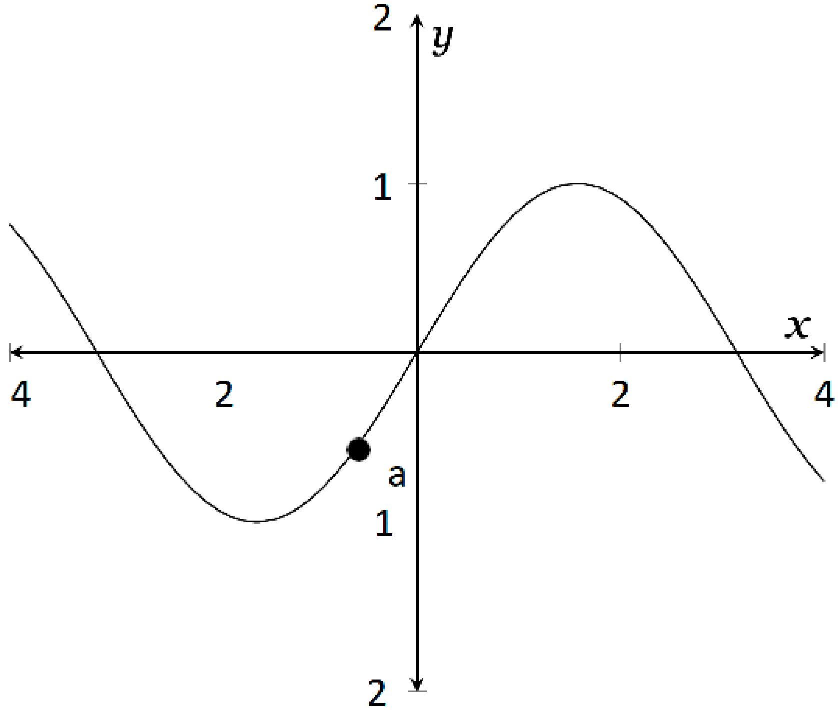
1

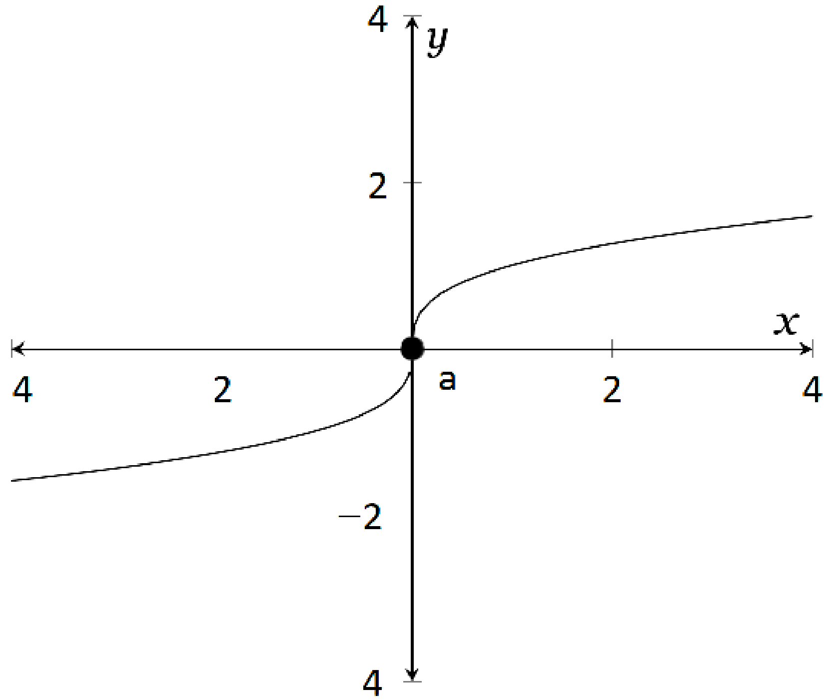


2



3





Is the green line a tangent to the function? Why or why not?

How would you define a tangent line?

Can a function have more than one tangent line at a single point? Why or why not? If so, can you draw an example of a function which has more than one tangent line at one of its points?

Decide if the following statement is true or false and explain your reasoning: A tangent line can only intersect a function at a single point.